# Analysis II Summary

Nicola Studer
nicstuder@student.ethz.ch

# 1 Ordinary differential equations

$$F(x, y^{(n)}, \dots, y'(x), y(x)) = 0$$

Given a function $F$ of $x, y$, where $y$ is a function itself. $F$ is an implicit ODE of **order** $n$.

**Linear ODE's**
$y^{(k)} + a_{k-1}(x)y^{(k-1)} + \dots + a_1(x)y' + a_0(x)y = b(x)$ with $a_{k-1}, \dots, a_0, b$ as cont. functions of $x$ in $I \subset \mathbb{R}$. If $\boldsymbol{b = 0}$ then the ODE is called **homogeneous**.

**Properties of linear ODEs**

1. all coefficients are continuous functions
2. no products of $y$ and its derivatives
3. no powers of $y$ and its derivatives
4. no functions which depend on $y$ or its derivatives
5. no leading coefficient in front of the highest derivative

**Thm** (Main result about linear ODEs)**.**

1. Let $\mathcal{S}_0$ be the set of solutions when $b = 0$. Then $\mathcal{S}_0$ is a vector space of dimension $k$. If $f_1, \dots, f_k$ are the solutions, then so is $a_1 f_1 + \dots a_k f_k$.
2. For any **initial condition** (i.e. for any $x_0 \in I$, $y \in \mathbb{C}^k$, $y = y_0, \dots, y_{k-1}$) there is a unique solution $f \in \mathcal{S}_0$ such that:
   $f(x_0) = y_0, f'(x_0) = y_1, \dots, f^{(k-1)}(x_0) = y_{k-1}$
3. For any arbitrary $b(x)$, the set of solutions of the ODE is $\mathcal{S}_b = \{f + f_p \mid f \in \mathcal{S}_0\}$ where $f_p$ is a particular solution of the ODE.
4. For any initial condition there is a unique solution $f \in \mathcal{S}_b$.

**Solve initial value problem**

1. Solve ODE
2. With initial values create LSE

## 1.1 Linear ODE of order 1

**Solution and derivation**
1. Solve the homogeneous ODE:

$$\begin{aligned} & y' + ay = 0 & \\ \implies \quad & y' = -ay & \\ \implies \quad & \tfrac{y'}{y} = -a & \text{(assume } y \neq 0 \text{ no } I) \\ \implies \quad & \ln(|y|) = -A + C & (A(x) = \textstyle\int a(x)\,dx) \\ \implies \quad & y = e^{-A+C} = z \cdot e^{-A} & \text{(simplify)} \end{aligned}$$

2. Find $f_p : I \to \mathbb{C}$ such that $f_p' + a(x)f_p = b(x)$ with variation of parameters or undetermined coefficients.
3. General solution: $f(x) = f_h(x) + f_p(x)$

### 1.1.1 Method of undetermined coefficients

| $\boldsymbol{b(x)}$ | **Guess** |
|---|---|
| $ae^{\alpha x}$ | $ce^{\alpha x}$ |
| $P_n(x)$ | $Q_n(x)$ |
| $a\sin(\beta x)$<br>$a\cos(\beta x)$ | $D\sin(\beta x) + E\cos(\beta x)$ |
| $ae^{\alpha x}\sin(\beta x)$<br>$ae^{\alpha x}\cos(\beta x)$ | $De^{\alpha x}\sin(\beta x) + Ee^{\alpha x}\cos(\beta x)$ |
| $P_n(x)e^{\alpha x}$ | $Q_n(x)e^{\alpha x}$ |
| $P_n(x)e^{\alpha x}\sin(\beta x)$<br>$P_n(x)e^{\alpha x}\cos(\beta x)$ | $e^{\alpha x}(Q_n(x)\sin(\beta x) + R_n(x)\cos(\beta x))$ |

1. If $b(x)$ is a linear combination of the basis functions, use corresponding linear combination of the functions.
2. If $f_p = f_0$, try to multiply it with $x^m$ where $m$ denotes the multiplicity of the eigenvalue.

**Variation of parameters**

1. Assume $f_p = z(x) \cdot e^{-A(x)}$ for a function $z : I \to \mathbb{C}$
2. Insert the equation and construct $z$:

$$\begin{aligned} & y' + ay = b \\ \implies \quad & z'e^{-A} = b \\ \implies \quad & z' = be^{A} \\ \implies \quad & z = \textstyle\int b(x)e^{A(x)}\,dx \\ \implies \quad & f_p = \textstyle\int b(t)e^{A(t)}\,dt \cdot e^{-A(x)} \end{aligned}$$

**Integration Factor**

$$\frac{dy}{dx} + a(x)y = b(x) \qquad (\dagger)$$

1. Multiply both sides of $(\dagger)$ with $e^{A(x)} = e^{\int a(x)\,dx}$
   $\frac{dy}{dx}e^{\int a(x)\,dx} + ya(x)e^{\int a(x)\,dx} = b(x)e^{\int a(x)\,dx}$
2. Observe the product rule on the left hand side:
   $\frac{d}{dx}ye^{\int a(x)\,dx} = b(x)e^{\int a(x)\,dx}$
3. Call $ye^{\int a(x)\,dx} := z(x) \implies y = z(x)e^{-A(x)}$ $(\ddagger)$
   $\frac{d}{dx}z(x) = b(x)e^{\int a(x)\,dx}$
4. Solve for $z(x)$: $z(x) = \int b(x)e^{A(x)}\,dx$
5. Insert $(\ddagger)$: $y = \left(\int b(x)e^{A(x)}\,dx\right)e^{-A(x)}$

## 1.2 Linear ODE with constant coefficients

$$Dy = b(x) \quad D = \frac{d^k}{dx^k} + a_{k-1}\frac{d^{k-1}}{dx^{k-1}} + \dots + a_0$$

**1. Solve homogeneous equation**

Assume $y = e^{\lambda x}$ for some $\lambda \in \mathbb{C}$. We put that guess in the initial formula and get the following (simplified) form:

$$e^{\lambda x}(\lambda^k + a_{k-1}\lambda^{k-1} + a_{k-2}\lambda^{k-2} + \dots + a_0) = e^{\lambda x} \cdot P(\lambda) = 0$$

Since $e^{\lambda x}$ can never be $0 \implies P(\lambda) = 0$. $P(\lambda)$ is the **characteristic polynomial** with its roots called **eigenvalues**.
**Thm.** $De^{\lambda x} = 0 \iff \lambda$ is a root of $P_D(\lambda)$

**Solutions**
The functions $f_{i,l} : x \mapsto x^l e^{\lambda_i x}$ span the solution space $S_0$ with $0 \leq l < m$, $m$ as the multiplicity of $\lambda_i$.

- If $\lambda = a + ib$ is EV of $P(\lambda)$, then $P(\overline{\lambda})$ is an EV.
- Complex root: $e^{(a+bi)\cdot x} = e^{ax}[\cos(bx) + i\sin(bx)]$
- If $b = e^{\alpha x}$, but $\alpha$ is a root of $P(\lambda)$ with $m = k$, then try $zx^k \cdot e^{\alpha x}$ for $y_p$

**Superposition Principle**
$D(y_1 + y_2) = D(y_1) + D(y_2) = b_1 + b_2$

**Separation of variables**
ODE separable if $\frac{dy}{dx} = b(x)g(y) \implies \frac{dy}{g(y)} = b(x)\,dx$. If $g(y) = 0$, then $y = y_h$ otherwise integrate both sides.

# 2 Differential calculus in $\mathbb{R}^n$

## 2.1 Terminology

| | |
|---|---|
| **Vector Field** | $f:\mathbb{R}^n \to \mathbb{R}^m \quad (m>1)$ |
| **Scalar Field** | $f:\mathbb{R}^n \to \mathbb{R}$ |
| **Monomial** | $f:\begin{cases}\mathbb{R}^n \to \mathbb{R} \\ (x_1,x_2,\dots,x_n)\mapsto \alpha x_1^{d_1}x_2^{d_2}\dots x_n^{d_n}\end{cases}$ |
| **Linear Map** | $f:\begin{cases}\mathbb{R}^n \to \mathbb{R} \\ x\mapsto Ax \quad (A\in\mathbb{C}^{m\times n})\end{cases}$ |
| **Affine Map** | $f:\begin{cases}\mathbb{R}^n \to \mathbb{R} \\ x\mapsto Ax+y_p \quad (y_p\in\mathbb{R}^m)\end{cases}$ |
| **Cart. Prod.** | $f:\begin{cases}\mathbb{R}^n \to \mathbb{R}^{s+t} \\ x\mapsto (f_1(x),f_2(x))\end{cases}$ |

**Converges of sequences**
$(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^n,\ y\in\mathbb{R}^n.\ \lim_{k\to\infty} x_k = y$

- $\Leftrightarrow$ $\forall\epsilon>0\,\exists N\geq 1\,\forall k\geq N: ||x_k-y||<\epsilon$
- $\Leftrightarrow$ For each $i$, $1\leq i\leq n$ the sequence $(x_{k,i})\subset\mathbb{R}$ of real numbers converges to $y_i\in\mathbb{R}$
- $\Leftrightarrow$ The sequence of real numbers $||x_k-y||\to 0$

**Def.** $f:X\subset\mathbb{R}^n\to\mathbb{R}^m$, $x_0\in X$. $f$ **has a limit** $y\in\mathbb{R}^m$ as $x\to x_0$ (with $x\neq x_0$) if

1. $\forall\epsilon>0\,\exists\delta>0\,\forall x\in X, x\neq x_0: ||f(x)-y||<\epsilon$
2. $\forall$ sequences $(x_k)$ in $X$ with $\lim x_k = x_0$ and $x_k\neq x_0$ converges the sequence $f(x_k)$ to $y$.

**Continuity**
$f:X\to\mathbb{R}^m$ cont. at $x_0$ if

1. $\forall\epsilon>0\,\exists\delta>0\,\forall x\in X:$
$$||x-x_0||<\delta \implies ||f(x)-f(x_0)||<\epsilon$$
2. $\forall$seq. $(x_k)$ with $\lim x_k = x_0 : \lim f(x_k) = f(x_0)$

$f$ cont. on $X$ if it is cont. $\forall x_0\in X$.

**Cor.**
1. *$f_1:\mathbb{R}^n\to\mathbb{R}^m, f_2:\mathbb{R}^n\to\mathbb{R}^s$ cont., then $f:(f_1,f_2):\mathbb{R}^n\to\mathbb{R}^{m+s}, x\mapsto(f_1(x),f_2(x))$ is cont.*
2. *$f:\mathbb{R}^n\to\mathbb{R}^m, x\mapsto(f_1(x),f_2(x),\dots)$ cont. $\iff \forall 1\leq i\leq m\ f_i:\mathbb{R}^n\to\mathbb{R}$ are cont.*
3. *$f:\mathbb{R}^n\to\mathbb{R}^m, x\mapsto Ax$ and polynomials are cont.*
4. *Sums/products of cont. functions are cont.*
5. *Functions with separated variables are cont. if each variable is cont.*
6. *Composition of cont. functions are cont.*
7. *If $f:\mathbb{R}^2\to\mathbb{R}$ is cont. Fix $y_0\in\mathbb{R}$. Define $g_{y_0}(x):=f(x,y_0)$. Then $g_{y_0}:\mathbb{R}\to\mathbb{R}$ is cont.*
$$\not\Rightarrow f:\mathbb{R}^2\to\mathbb{R} \text{ is cont.}$$

**Sandwich lemma**
$f,g,h:\mathbb{R}^n\to\mathbb{R}$, $\forall x\in\mathbb{R}^n: f(x)\leq g(x)\leq h(x)$
$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x) \implies \lim_{x\to a} g(x) = L$$

**Polar Coordinates**
For $f:\mathbb{R}^2\to\mathbb{R}$ polar coordinates are sometimes helpful. $p=r\cos\theta \quad q=r\sin\theta$
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(r\cos\theta,r\sin\theta)\to(0,0)} f(p,q) = \ldots = \lim_{r\to 0}\zeta$$

## 2.2 Sets Bounds $M\subseteq\mathbb{R}^n$

$M$ **is bounded** $\overset{\text{def}}{\iff}$ $\{||x||\in\mathbb{R}\mid x\in M\}$ is bounded

$M$ **is open** $\overset{\text{def}}{\iff}$ $\forall p\in M:\exists r\in\mathbb{R}^{>0}: B_p(r)\subseteq M$
$\overset{\text{def}}{\iff}$ $\mathbb{R}^n\setminus M$ is closed

$M$ **is closed** $\overset{\text{def}}{\iff}$ $\forall (x_k)_{k\in\mathbb{N}}\subseteq M$ that converge to $x\in\mathbb{R}^n: x\in M$

$M$ **is compact** $\overset{\text{def}}{\iff}$ $M$ closed and bounded

**Special Sets**
- $\mathbb{R}^n$ and $\emptyset$ are the **only** open and closed sets of $\mathbb{R}^n$.
- The open disc $B_r(x_0)=\{x\in\mathbb{R}^n\mid |x-x_0|<r\}$ is bounded and open.
- The closed disc $\overline{B_r(x_0)}=\{x\in\mathbb{R}^n\mid |x-x_0|\leq r\}$ is closed.
- $I_1\times\ldots I_n$ is closed (compact) if each interval $I_i$ is closed (compact)

**Thm.** $f:\mathbb{R}^n\to\mathbb{R}^m$ cont. $\forall Y\subseteq\mathbb{R}^m$ **closed**, the set $f^{-1}(Y)=\{x\in\mathbb{R}^n\mid f(x)\in Y\}$ is closed.

**Thm.** $f:\mathbb{R}^n\to\mathbb{R}^m$ cont. $\forall Y\subseteq\mathbb{R}^m$ **open**, the set $f^{-1}(Y)=\{x\in\mathbb{R}^n\mid f(x)\in Y\}$ is open.

**Min-Max theorem**
$X\subseteq\mathbb{R}^n$ compact. $f:X\to\mathbb{R}$ cont. $\implies$
$$\exists x_+,x_-\in X: f(x_+)=\sup_{x\in X}(f(x)),\quad f(x_-)=\inf_{x\in X} f(x)$$

## 2.3 Partial derivatives

**Def.** $f:X\subseteq\mathbb{R}^n\to\mathbb{R}$, $X$ open. The **partial derivative** of $f$ with respect to $x_i$ at the point $a\in\mathbb{R}^n$ is
$$\frac{\partial f}{\partial x_i}(a) = \lim_{h\to 0}\frac{f(a+he_i)-f(a)}{h} \quad ((e_i)_j=\delta_{ji}, j=1,\dots,n)$$

If $f:X\to\mathbb{R}^m$ for $x_0\in\mathbb{R}^n$, then
$$\frac{\partial f}{\partial x_i}(a) = \begin{bmatrix}\frac{\partial f_1}{\partial x_i}(a)\\ \vdots \\ \frac{\partial f_m}{\partial x_i}(a)\end{bmatrix}$$

**Cor.** *$X\subseteq\mathbb{R}^n$ open, $f,g:X\to\mathbb{R}^m$:*
- $\partial_{x_i}(f+g)=\partial_{x_i}(f)+\partial_{x_i}(g)$
- $\partial_{x_i}(f\cdot g)=\partial_{x_i}(f)\cdot g+f\cdot\partial_{x_i}(g)$ *if* $m=1$
- $\partial_{x_i}(f/g)=(\partial_{x_i}(f)\cdot g-f\cdot\partial_{x_i}(g))/g^2$ *if* $m=1, g\not\equiv 0$

**Def.** The **Jacobi Matrix** of $f:X\subset\mathbb{R}^n\to\mathbb{R}^M$ at $x\in X$:
$$\mathcal{J}_f(x) = \left[\frac{\partial f_i}{\partial x_j}(x)\right]_{\substack{1\leq i\leq m\\1\leq j\leq n}}$$

**Def.** $f:X\subseteq\mathbb{R}^n\to\mathbb{R}$, $X$ open. The **gradient** of $f$:
$$\nabla f(x) = \mathcal{J}_f(x)^\top$$

The gradient points in the direction of greatest increase and is perpendicular to the level set.

## 2.4 The differential

**Def.** $f:\mathbb{R}^n\to\mathbb{R}^m$ is diff. at $x_0$, with **differential** $u$, if there exists a linear map $u:\mathbb{R}^n\to\mathbb{R}^m$ such that
$$\lim_{\substack{x\to x_0\\x\neq x_0}}\frac{f(x)-(f(x_0)+u(x-x_0))}{||x-x_0||}=0$$

The linear map $u:\mathbb{R}^n\to\mathbb{R}^m$ is called (total) **differential** of $f$ at $x_0$, denoted by $df(x_0), d_{x_0}f$

**Thm.** $f, g : \mathbb{R}^n \to \mathbb{R}^m$ diff. at $x_0 \implies$

1. $f$ is cont. at $x_0$
2. $f$ admits partial derivatives on $x_0$ w.r.t each variable
3. $\mathcal{J}_f(x_0)$ is the differential w.r.t the standard basis.
4. $d_{x_0}(f \pm g) = d_{x_0} f \pm d_{x_0} g$
5. $d_{x_0}(f \cdot g) = (d_{x_0} f) g(x_0) + f(x_0) \cdot (d_{x_0} g)$ if $m = 1$
6. If $m = 1$ and $g \not\equiv 0$, the $f/g$ is diff.

**Multivaraible Chain Rule**
Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open $f : X \to Y, g : T \to \mathbb{R}^p$ diff functions, then

$$d_{x_0}(g \circ f) = d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$$

The Jacobian satisfies: $\mathcal{J}_{g \circ f}(x_0) = \mathcal{J}_g(f(x_0)) \cdot \mathcal{J}_f(x_0)$

**Partial Convergence**
If $f : X \to \mathbb{R}^m$ has all partial derivatives $\frac{\partial f_i}{\partial x_j} : X \to \mathbb{R}^m$ and if these functions are cont. in $X \implies f$ is diff. on $X$.

**Def.** The **tangent space** at $x_0$ is the graph of the affine linear map

$$g(x) = f(x_0) + (d_{x_0} f)(x - x_0) \text{ i.e } \{(x, g(x)) \in \mathbb{R}^n \times \mathbb{R}^m\}$$

**Def.** $X \subseteq \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$, $v \in \mathbb{R}^n \neq 0$, $x_0 \in X$. The **directional derivative** in direction $v$ is
$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \frac{d}{dt} f(x_0 + tv)\Big|_{t=0} = d_v f(x_0) = \mathcal{J}_f(x_0) \cdot v$

**Summed up**

- $f$ differentiable $\implies f$ continuous
- $f$ has all partial derivatives $\not\Longrightarrow f$ continuous

## 2.5 Higher order partial derivatives

**Def.** $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^m$. We say $f$ is diff. of class $C^1$ if $f$ is diff. on $X$ and all its partial derivatives are continuous.

The set of all $C^1$ functions are denoted by $C^1(X; \mathbb{R}^m)$.

Let $k \geq 2$, then $f \in C^k(X; R^m)$ if its diff. and each $\partial_{x_i} f \in C^{k-1}(X; \mathbb{R}^m)$.

$f$ is smooth or $C^\infty$ if $f \in C^k(X; \mathbb{R}^m)$ $\forall k$.

**Known $C^\infty$ functions**
All polynomials, trigonometric and exponential functions

**Mixed derivatives commute**
If $f \in C^k$, $k \geq 2$ then the partial derivatives of oder $\leq k$ are independent of the order of differentiation.

$$\frac{\partial}{\partial x_{i_k}} \cdots \left( \frac{\partial}{\partial x_{i_2}} \left( \frac{\partial f}{\partial x_{i_1}} \right) \right) = \frac{\partial^k f}{\partial x_{i_k} \cdot \ldots \cdot \partial x_{i_2} \cdot \partial x_{x_{i_1}}}$$

**Def** (Hessian)**.** $f : X \to \mathbb{R}, X \subset \mathbb{R}^n$. If $f \in C^2(X; \mathbb{R})$, $x_0 \in X$ the Hessian matrix of $f$ at $x$ is the symmetric square matrix

$$\text{Hess}_f(x_0) = \nabla^2 f(x_0) = \left[ \frac{\partial^2 f(x_0)}{\partial x_i \partial x_j} \right]_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$$

**Taylor Polynomial for $f : \mathbb{R}^n \to \mathbb{R}$**
$\approx f(y)$ for $y$ close to $x_0$. To calculate use $y = x - x_0$.

$$\begin{aligned} T_1 f(x_0; y) &= f(x_0) + \nabla f(x_0) \cdot y \\ &= f(x_0) + \frac{\partial f}{\partial x_1}(x_0) y_1 + \ldots \frac{\partial f}{\partial x_n}(x_0) y_n \end{aligned}$$

$$\begin{aligned} T_2 f(x_0; y) =& f(x_0) + \nabla f(x_0) \cdot y \\ &+ \frac{1}{2!} y \cdot \text{Hess}_f(x_0) \cdot y^\top \end{aligned}$$

$$T_k f(x_0; y) = f(x_0) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0) y_i + \ldots + \sum_{m_1 + \ldots m_n = k} \frac{1}{m_1! \cdot m_2! \cdot m_n!} \frac{\partial^k f}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}}(x_0) \cdot y_1^{m_1 \ldots y_n^{m_n}}$$

**Taylor Polynomials w/ Einstein Sum Convention**
$T_k f(x_0; y) = f(x_0) + (\partial_i f)(x_0) y_i + \frac{1}{2!}(\partial_{ij} f)(x_0) y_i y_j + \frac{1}{3!}(\partial_{ijk} f) y_i y_j y_k + \cdots$

**Taylor Approximation**
Let $f \in C^k(X; \mathbb{R}), x_0 \in X$

$$f(x) = T_k f(x_0, x - x_0) + E_k(f, x, x_0)$$

which implies

$$\lim_{x \to x_0} \frac{E_k(f, x, x_0)}{||x - x_0||^2} = 0$$

## 2.6 Critical points

**Def.** $x_0 \in X$ of $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is a **local** maximum if there is a neighborhood $B_{x_0}(r) := \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\} \subseteq X$ such that $\forall x \in B_{x_0}(r) : f(x) \leq f(x_0)$. Vice versa for minimum.

**Def.** $x \in X$ is called a **critical point** of $f$ if $\nabla f(x_0) = 0$. These are candidates for local extrema. A critical point which is not an extrema is called **saddle point**.

**Thm.** $f : X \subset R^n \to \mathbb{R}$ diff. on the interior of $X$ and $X$ closed and bounded, then a **global** extrema of $f$ exists and is either at a point $x_0 \in$ interior of $X$ for which $\nabla f(x_0) = 0$ or $x_0 \in$ boundary of $x$.

**Def** (Non-degenerate critical point of $f \in C^2(X, \mathbb{R})$)**.**

$$\det(\text{Hess}_f(x_0)) \neq 0$$

**Special case for degenerate critical points**
If $\nabla f(x_0) = 0$, but also $\det \text{Hess}_f(x_0) = 0$, then we have to calculate each case individually.

**Thm.** $f : X \subset \mathbb{R}^n \to \mathbb{R}$, $f \in C^2(X, \mathbb{R})$. Let $x_0 \in X$ be a critical point of $f, \nabla f(x_0) = 0$. Then

1. $\text{Hess}_f(x_0)$ pos. def. $\implies$ loc. min.
2. $\text{Hess}_f(x_0)$ neg. def. $\implies$ loc. max.
3. $\text{Hess}_f(x_0)$ indefinite $\implies$ saddle point.

**Definiteness of matrices**
A matrix $A$ is **positive definite**
$\iff xAx^\top > 0 \quad \forall x \in \mathbb{R}^n$
$\iff$ all eigenvalues of $A$ are positive
$\iff$ all principal minors of $A$ are positive:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

1. $a > 0$
2. $ad - b^2 > 0$
3. $\det(A) > 0$

$A$ is **negative definite** $\iff -A$ positive definite
$A$ is **indefinite** $\iff$ $A$ neither pos. semi- nor neg. semi-def.

**Cor** (Closed form expression for $3 \times 3$ matrix)**.**

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

## 2.7 Change of variables

**Def** (Change of variables)**.** $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$ diff. $f$ is a change of variables around $x_0$ if there is a radius $r > 0$, such that the restriction of $f$ to the ball $B_r(x_0) := \{x \in \mathbb{R}^n \mid ||x - x_0|| < r\}$ has the property that the image $Y = f(B_r(x_0))$ is open in $\mathbb{R}^n$ and there exists a differentiable map $g : Y \to B$ s.t. $f \circ g = id = g \circ f$.

**Inveres function theorem**
$X \subseteq \mathbb{R}^n$ open, $f : X \to \mathbb{R}^n$ diff. If $x_0 \in X$ is such that $\det(\mathcal{J}_f(x_0)) \neq 0$, then $f$ is a change of variables around $x_0$. Moreover the Jacobian of $g$ is determined by

$$\mathcal{J}_g(f(x_0)) = \mathcal{J}_f(x_0)^{-1}$$

Analogous of the fact that if $f' > 0$ (or $f' < 0$) for a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, then $f$ is bijective.

### 2.7.1 Important change of variables (coordinates)

1. **Polar** $f : \begin{cases} [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \\ (r, \theta) \mapsto (r\cos\theta, r\sin\theta)^\top \end{cases}$

   The Jacobian of the change of variable is given by:

$$\mathcal{J}_f(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \qquad \det \mathcal{J}_f(r, \theta) = r$$

2. **Cylindrical** $f : \begin{cases} [0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3 \\ (r, \theta, z) \mapsto (r\cos\theta, r\sin\theta, z)^\top \end{cases}$

   The Jacobian of the change of variable is given by:

$$\mathcal{J}_f(r, \theta, z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \det \mathcal{J}_f(r, \theta) = r$$

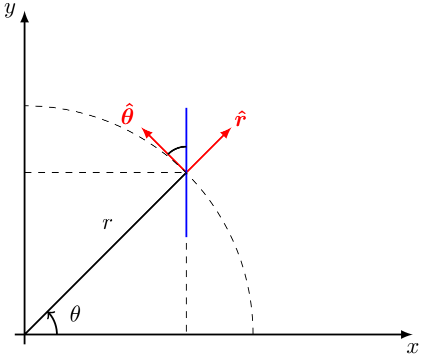
3. **Spherical** $f : \begin{cases} [0, \infty) \times [0, 2\pi) \times [0, \pi] \to \mathbb{R}^3 \\ (r, \theta, \varphi) \mapsto (r\cos\theta\sin\varphi, r\sin\theta\sin\varphi, r\cos\varphi) \end{cases}$

   The Jacobian of the change of variable is given by:

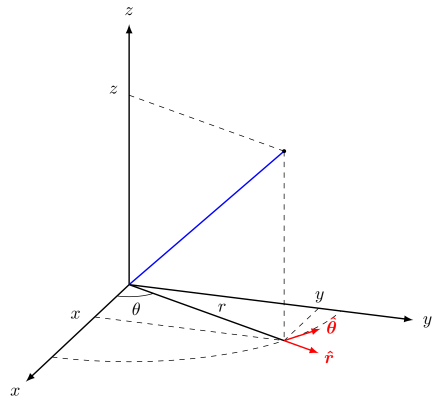
$$\mathcal{J}_f(r, \theta, \varphi) = \begin{bmatrix} \cos\theta\sin\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\varphi & 0 & -r\sin\varphi \end{bmatrix}$$

$$\det \mathcal{J}_f(r, \theta) = r^2 \sin\varphi$$

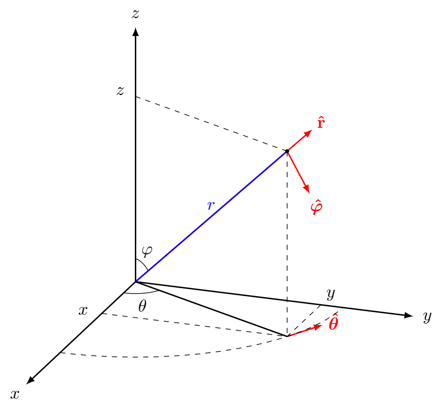
**Polar Coordinates**




**Cylindrical coordinates**




**Spherical Coordinates**




**Partial derivatives after a change of variable**
Consider $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ and a change of variables $g : U \to X$ that expresses the variables $(x_1, \dots, x_n)$ in terms of $(y_1, \dots, y_n)$, such that $x_i = g_i(y_1, \dots, y_n)$. Thus the composite $h = f \circ g : U \to \mathbb{R}$ is the function $f$ expressed in terms of the "new" variables $y$. By chain rule:

$$d_y h = df(g(y)) \circ dg(y) = \nabla f(x)^\top \circ dg(y) = (\partial_{y_1} h \ \cdots \ \partial_{y_n} h)$$

where we used that $df(g(y)) = df(x) = \nabla f(x)^\top$.

$$\partial_{y_i} h = \frac{\partial f}{\partial x_1}\frac{\partial g_1}{\partial y_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial g_n}{\partial y_i}$$

**Abuse of notation**

1. One thinks of $f$ and $h$ as being the same function, simply expressed in different coordinate systems. Thus

$$\partial y_i f = \partial_{x_1} f \, \partial_{y_i} g_1 + \ldots + \partial_{x_n} f \, \partial y_i g_n$$

2. One thinks of $g_i$ as being the variable $x_i$, expressed in terms of the new variables $y$. Thus

$$\partial y_i f = \partial_{x_1} f \, \partial_{y_i} x_1 + \ldots + \partial_{x_n} f \, \partial y_i x_n$$

**Change of variables for integration**
Let $\overline{X} \subseteq \mathbb{R}^n$, $\overline{Y} \subseteq \mathbb{R}^n$ be compact subsets. Let $\varphi : \overline{X} \to \overline{Y}$ be a continuous map with $\overline{X} = X \cup B$, $\overline{Y} = Y \cup C$ where $X$ and $Y$ are open, $B$ and $C$ negligible. The restriction $\varphi : X \to Y$ is a bijective map of class $C^1$ such that for all $x \in X$ it holds $\det \mathcal{J}_\varphi(x) \neq 0$. Assume $f : \overline{Y} \to \mathbb{R}$ then:

$$\int_{\overline{X}} f(\varphi(x)) \left|\det \mathcal{J}_\varphi(x)\right| \, dx = \int_{\overline{Y}} f(y) \, dy$$

**Shortcuts (substitutions from 2.7.1)**

- Polar Coordinates: $dx\,dy = r\,dr\,d\theta$
- Cylindrical coordinates: $dx\,dy\,dz = r\,dr\,d\theta\,dz$
- Spherical coordinates: $dx\,dy\,dz = r^2\sin(\varphi)\,dr\,d\theta\,d\varphi$

**Example**: Let $X$ be a quarter circle and $z = \frac{1}{1+x^2+y^2}$:

$$\iint_X \frac{dx\,dy}{1+x^2+y^2} = \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r^2} \cdot r\,dr\,d\theta$$

# 3 Integration in $\mathbb{R}^n$

**Def.** $\int_a^b f(x)\,dx = \begin{bmatrix} \int_a^b f_1(x)\,dx \\ \vdots \\ \int_a^b f_n(x)\,dx \end{bmatrix}$ for $f : \mathbb{R} \to \mathbb{R}^n$

## 3.1 Line Integrals

**Def.** A **parameterized curve** $\gamma : [a,b] \to \mathbb{R}^n$ is a continuous map and piecewise in $C^1$ i.e. $\exists k > 1$ and a partition $a = t_0 < t_1 < \ldots < t_k = b$ s.t. $\gamma\big|_{]t_{j-1},t_j[}$ is $C^1$ for $1 \leq j \leq k$. $\gamma(t)$ is a parameterization of the curve $\text{Im}\gamma = \gamma([a,b])$.

**Def.** Let $\gamma : [a,b] \to \mathbb{R}^n$ be a parameterized curve in $\mathbb{R}^n$. $X \subset \mathbb{R}^n$ a subset of $\mathbb{R}^n$ which contains the image of $\gamma$. $f : X \to \mathbb{R}^n$ a continuous function. The integral

$$\int_a^b \langle f(\gamma(t)), \gamma'(t)\rangle\,dt \quad \text{denoted} \quad \int_\gamma f(s)\cdot ds$$

is called the line or path integral of $f$ along $\gamma$.

**Def.** Let $\gamma : [a,b] \to \mathbb{R}^n$ be a parameterized curve. An **oriented reparameterization** of $\gamma$ is a parameterized curve $\sigma : [c,d] \to \mathbb{R}^n$ such that $\sigma = \gamma \circ \varphi$, where $\varphi : [c,d] \to [a,b]$ is a cont. map, diff. on $]a,b[$ that is strictly increasing and satisfies $\varphi(a) = c$ and $\varphi(b) = d$. Also $\gamma = \sigma \circ \varphi^{-1}$

**Properties of the line intergral**

1. Only dependent on the image of the curve $\gamma$. If $\sigma$ is an oriented reparameterization of $\gamma$, then
$$\int_\gamma f(s)\cdot ds = \int_\sigma f(s)\cdot ds$$
2. Let $\gamma_1 + \gamma_2$ be the concatenation of these two curves.
$$(\gamma_1+\gamma_2)(t) := \begin{cases} \gamma_1(t) & t \in [a,b] \\ \gamma_2(t-b+c) & t \in [b, d+b-c] \end{cases}$$
$$\int_{\gamma_1+\gamma_2} f(s)\cdot ds = \int_{\gamma_1} f(s)\cdot ds + \int_{\gamma_2} f(s)\cdot ds$$
3. Let $-\gamma : t \mapsto \gamma(a+b-t)$ be the line in opposite direction
$$\int_{-\gamma} f(s)\cdot ds = -\int_\gamma f(s)\cdot ds$$

**Def.** A differentiable function $g : X \subset \mathbb{R}^n \to \mathbb{R}$, such that $\nabla g = f, f : X \to \mathbb{R}^n$ is called a **potential** for $f$.

**Usefulness of potentials**
Let $g$ be a potential of $f$, then

$$\int_\gamma f(s)\cdot ds = \int_\gamma \nabla g(s)\cdot ds = g(\gamma(b)) - g(\gamma(a)).$$

Thus the path integral of $f$ only depends on the values of $g$ at the end points of the curve.

**Def.** Let $X \subseteq \mathbb{R}^n$ and $f : X \to \mathbb{R}^n$ a continuous vector field. If for any $x_1, x_2 \in X$ the line integral $\int_\gamma f(s)\cdot ds$ is independent of the choice of the curve $\gamma$, then $f$ is called **conservative**.

**Important equivalences**
$f : X \to \mathbb{R}^n$ is conservative
$\overset{\text{def}}{\iff}$ The line integral of $f$ is independent of the path
$\iff$ $f = \nabla g$ for a $g : X \to \mathbb{R}$ (i.e. $f$ has a potential)
$\iff$ $\int_\gamma f(s)\cdot ds = 0$ for any closed $\gamma$

**Thm.** $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^n, C^1$ vector field and $X$ open.

$$f \text{ conservative} \implies \frac{\partial f_j}{\partial x_i} = \frac{\partial f_i}{\partial x_j} \text{ for } 1 \leq i,j \leq n$$

**Def.** $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3, C^1$, then the curl of $f$ is defined as

$$\text{curl}(f) := \begin{pmatrix} \partial_y f_3 - \partial_z f_2 \\ \partial_z f_1 - \partial_x f_3 \\ \partial_x f_2 - \partial_y f_1 \end{pmatrix}$$

**Thm.** $f : \mathbb{R}^3 \to \mathbb{R}^3$ $f$ conservative $\implies \text{curl}(f) = 0$

**Def.** A subset $X \subseteq \mathbb{R}^n$ is **star shaped** if $\exists x_0 \in X$ such that $\forall x \in X$ the line segment of $x$ to $x_0$ is contained in $X$.

**Def.** A subset $X \subseteq \mathbb{R}^n$ is convex, when for any $x, y \in X$ the line segment from $x$ to $y$ is contained in $X$.

$$\text{convex} \implies \text{star-shaped}$$

**Thm.** If $X$ star-shaped open and $f \in C^1$ vector field. Then

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}\ \forall 1 \leq i,j \leq n \implies f \text{ is conservative}$$
$$\text{curl}(f) = 0 \implies f \text{ is conservative}$$

## 3.2 Riemann Integrals

**Def.** A **rectangle** in $\mathbb{R}^n$ is a product

$$Q := [a_1,b_1] \times \ldots \times [a_n,b_n] = \prod_{j=1}^n I_i$$

of $n$ intervals $I_i = [a_i,b_i]$ (not necessarily closed), and

$$\text{vol}(Q) := \int_Q 1\,dx = \prod_{i=1}^n (b_i - a_i).$$

**Def.** Let $P$ be a partition (collection) of $Q = Q_1, \ldots, Q_k$ s.t. $Q = \bigcup_{i=1}^k Q_i$ and all $Q_i$ are pairwise disjoint and consider $f : \mathbb{R}^n \to \mathbb{R}$. The upper/lower Riemann sum are defined as:

$$L(P,f) = \sum_{j=1}^k (\inf_{Q_j} f)\cdot\text{vol}(Q_j),\ U(P,f) = \sum_{j=1}^k (\sup_{Q_j} f)\cdot\text{vol}(Q_j)$$

and we define the lower and upper Riemann integral as

$$\underline{\text{I}}(f) := \sup_P\{L(P,f)\},\ \bar{\text{I}}(f) = \inf_P\{U(P,f)\}$$

**Def.** $f : Q \to \mathbb{R}$ is called **integrable** if $\underline{\text{I}}(f) = \bar{\text{I}}(f)$.

$$\int_Q f\,dx = \int_Q f(x_1,\ldots,x_n)\,dx_1\ldots dx_n$$

**Thm.** $f$ is cont. and bounded on $Q$, then $f$ is integrable.

**Properties**
$f, g : Q \subseteq \mathbb{R}^n \to \mathbb{R}$ integrable and $\alpha, \beta \in \mathbb{R}$

1. **Linearity**: $\int_Q (\alpha f + \beta g)\,dx = \alpha\int_Q f\,dx + \beta\int_Q g\,dx$
2. **Positivity**: If $f \leq g$, then $\int_Q f\,dx \leq \int_Q g\,dx$
3. **Upper bound**: $|\int_Q f\,dx| \leq \int_Q |f|\,dx$
4. **Zero**: If $f \geq 0$, then $\int_Q f\,dx \geq 0$
5. **Triangle Ine.:** $|\int_Q (f+g)\,dx| \leq \int_Q |f|\,dx + \int_Q |g|\,dx$
6. **Domain additivity**: If $X_1$ and $X_2$ are compact subsets of $\mathbb{R}^n$ and $f$ is continuous on $X_1 \cup X_2$, then
$$\int_{X_1\cup X_2} f\,dx = \int_{X_1} f\,dx + \int_{X_2} f\,dx - \int_{X_1\cap X_2} f\,dx$$

**Fubini's theorem**

If $Q = [a_1, b_1] \times \ldots \times [a_n, b_n]$, $f : Q \to \mathbb{R}$ cont, then

$$\int_Q f(x)\,dx = \int_{a_1}^{b_1} \left( \int_{a_2}^{b_2} \cdots \left( \int_{a_n}^{b_n} f(x)\,dx_n \right) \ldots dx_2 \right) dx_1$$

The order of integration is irrelevant.

**Fubini's theorem for general regions**

$X \subseteq \mathbb{R}_n$, $f : X \to \mathbb{R}$, $n_1, n_2 \geq 1$ and $n = n_1 + n_2$, then for $x \in \mathbb{R}^n = (x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2})$, define

$$X_{x_1} := \{x_2 \in \mathbb{R}^{n_2} \mid (x_1, x_2) \in X\} \subseteq \mathbb{R}^{n_2}$$
$$X_1 := \{x_1 \in \mathbb{R}^{n_1} \mid X_{x_1} \neq \emptyset\} \subseteq \mathbb{R}^{n_1}$$

If $g(x_1) := \int_{X_{x_1}} f(x_1, x_2)\,dx_2$ is continuous on $X_1$, then

$$\int_X f(x)\,dx = \int_{X_1} g(x_1)\,dx_1 = \int_{X_1} \int_{X_{x_1}} f(x_1, x_2)\,dx_2\,dx_1$$

**Integrals with separated variables**

Suppose $X = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$, and $f$ is a function with separated variables given by $f(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$ where each function $f_i$ is continuous (so $f$ is continuous). Then:

$\int_X f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n = \left( \int_{a_1}^{b_1} f_1(x)\,dx \right) \cdots \left( \int_{a_n}^{b_n} f_n(x)\,dx \right)$

**Def.** For $1 \leq m \leq n$ a $m$-parameterized set or parameterized $m$-set is a continuous function $\varphi : [a_1, b_1] \times \ldots \times [a_m, b_m] \to \mathbb{R}^n$ which is $C^1$ on $(a_1, b_1) \times \ldots \times (a_m, b_m)$. If $m = 1$ then $\varphi$ is a parameterized curve in $\mathbb{R}^n$.

**Def.** A set $Y \subseteq \mathbb{R}^n$ is called negligible if $\exists$finitely many $\varphi_i$, parameterized $m_i$-sets with $m_i \leq n$ such that $1 \leq i \leq k$

$$Y \subseteq \bigcup_{i=1}^{k} \varphi_i(x_i)$$

where $\varphi_i : x_i \to \mathbb{R}^n$

**Thm.** If $Y \subseteq \mathbb{R}^n$ is negligible closed bounded then

$$\int_Y f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n = 0$$

$\forall f : Y \to \mathbb{R}$ continuous

## 3.3 Improper Integrals

**Def.** We say $f$ is integrable on $I \times J$ if

$$\lim_{b \to \infty} \int_a^b \int_I f(x, y)\,dx\,dy = \lim_{b \to \infty} \int_I \int_a^b f\,dy\,dx$$

exists and denote the limit with

$$\int_a^\infty \int_I f\,dx\,dy = \int_{I \times J} f\,dx\,dy$$

**Def.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a non compact set and $f$ a function such that $\int_K f\,dx$ exists for every compact set $K \subset X$ and suppose $f \geq 0$. Consider the sequence $X_k$ $k = 1, 2, \ldots$ s.t.

1. $X_k \subset X$ bounded and closed
2. $X_k \subseteq X_{k+1}$
3. $\bigcup_{k=1}^\infty X_k = X$

Then if the following limit exists, the integral converges:

$$\int_X f\,dx := \lim_{n \to \infty} \int_{X_n} f\,dx$$

## 3.4 The Green formula

**Def.** A **simple closed parameterized curve** $\gamma : [a, b] \to \mathbb{R}^2$ is a closed parameterized curve such that $\gamma(t) \neq \gamma(s)$ unless $t = s$ or $\{s, t\} = \{a, b\}$ and such that $\gamma'(t) \neq 0$ for $a < t < b$. If $\gamma$ is only piecewise $C^1$ inside $]a, b[$, this condition only applies where $\gamma'(t)$ exists.

**Green's Theorem**

Let $f : X \to \mathbb{R}^2$ $C^1$ vector field, $X$ closed and bounded where $\partial X = \bigcup_{i=1}^n \gamma_i$ union of simple closed curves so that $X$ is always to the left of the curve $\gamma = \bigcup_{i=1}^n \gamma_i$ then

$$\iint_X \left( \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx\,dy = \sum_{i=1}^n \int_{\gamma_i} f \cdot ds = \int_\gamma f \cdot ds$$

**Both directions**

1. Use the double integral to calculate the line integral
2. Use the line integral to calculate a double integral

## General tips and tricks

**How to find the potential of a function**

Let $h := g(x_1, \ldots, x_n)$ and $\nabla g = f$. To find $g$, construct the following system of equation:

(1) $\partial_{x_1} g = f_1(x_1, \ldots, x_n) \iff h = \int f_1(x_1, \ldots x_n)\,dx_1$

(2) $\partial_{x_2} g = f_2(x_1, \ldots, x_n) \implies \partial_{x_2} h = f_2(x_1, \ldots, x_n)$

$\vdots$

($n$) $\partial_{x_n} g = f_n(x_1, \ldots x_n) \implies \partial_{x_n} h = f_n(x_1, \ldots, x_n)$

When integrating $f_1(x_1, \ldots, x_n)$ do not forget to carry a function $\tilde{z}(x_2, \ldots, x_n)$ depending only on $x_2, \ldots, x_n$. With the other conditions it is possible to find a unique $\tilde{z}$.

**How to find global maxima/minima**

1. Find the candidates in the interior
2. Bounded $\Rightarrow$ Use parameterization ($\gamma$) of the bound. To calculate the candidates use $g := f(\gamma(t))$ and $g'$.
3. Evaluate candidates + all corners of the bound

**Cor** (Inverse $2 \times 2$ matrix)**.**

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Taylor polynomials with $\mathcal{O}$-Notation**

**Example 1**: Compute Taylor polynomial of order 2 of $f(x,y,z) = \cos\left(\frac{x}{1+y^2} - \frac{y}{1+z^2}\right)$ at $(x,y,z) = (0,0)$. We can use $\cos(t) = 1 - \frac{t^2}{2} + \mathcal{O}(t^4)$. Thus $f(x,y,z)$

$$= 1 - \frac{1}{2}\left(\frac{x^2}{(1+y^2)^2} + \frac{y^2}{(1+z^2)^2} - \frac{2xy}{(1+y^2)(1+z^2)}\right) + \mathcal{O}((x^2+y^2+z^2)^2)$$
$$= 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + xy + \mathcal{O}((x^2+y^2+z^2)^2)$$

And thus $T_2 f((0,0,0),(x,y,z)) = 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2 + xy$.

**Example 2**: Compute Taylor polynomial of order 2 of $f(x,y,z) = 2\exp\left(x + y^2 + z^3\right)$ at $(x,y,z) = (0,0,0)$. We can use $e^t = 1 + t + \frac{t^2}{2} + \mathcal{O}(t^3)$. Thus $f(x,y,z)$

$$= 2(1 + x + y^2 + z^3 + \frac{1}{2}(x+y^2+z^3)^2 + \mathcal{O}(|(x,y,z)^3|))$$
$$= 2 + 2x + x^2 + 2y^2 + \mathcal{O}(|(x,y,z)|^3)$$

And thus $T_2 f((0,0,0),(x,y,z)) = 2 + 2x + x^2 + 2y^2$.

# Analysis I Stuff

**Derivative rules**

Linearity: $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$

Product rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$

Chain rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Inverse: $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$, $y_0 = f(x_0)$

## Trigonometric functions

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

**Useful bound for** sin:$\forall x \in \mathbb{R}_0^+ : \sin(x) \leq x$

*Proof.* Let $g(x) = x - \sin(x)$ with $g'(x) = 1 - \cos(x) \geq 0$ □

## Natural Logarithm Rules

| | |
|---|---|
| $\ln(1) = 0$ | $\ln(e) = 1$ |
| $\ln(xy) = \ln(x) + \ln(y)$ | $\ln(x/y) = \ln(x) - \ln(y)$ |
| $\ln(x^y) = y \cdot \ln(x)$ | $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$ |
| $(x^\alpha)^\beta = x^{\alpha\cdot\beta}$ | $\frac{x-1}{x} \leq \ln(x) \leq x - 1$ |
| $\ln(1 + x^\alpha) \leq \alpha x$ | $\log_\alpha(x) = \frac{\ln(x)}{\ln(\alpha)}$ |

## Function Properties

Consider an arbitrary function $f : X \to Y$.

**Def** (Well defined)**.** $f$ is well defined if $f(x)$ exists $\forall x \in X$.

**Def** (Injective)**.** $\forall x, y \in X : f(x) = f(y) \implies x = y$

- Assume $f(x) = f(y)$ and then show that $x = y$
- Assume $x \neq y$ and show that $f(x) \neq f(y)$

**Def** (Surjective)**.** $\forall y \in Y\ \exists x \in X : f(x) = y$

- Take arbitrary $y \in Y$ and show that there is an element $x \in X$. Consider $f(x) = y$ and solve for $x$ and check whether or not $x \in X$.

**Def** (Bijective)**.** $f$ injective and surjective $\implies f$ bijective

## Integration Methods

**Partial Inegration**

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$
$$\int_a^b f(x)g'(x)\,dx = (f \cdot g)|_a^b - \int_a^b f'(x)g(x)\,dx$$
$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

- Choose $g'$: exp → trig → poly → inverse trig. → logs
- Choose $f$: logs → inverse trig. → poly → trig → exp
- Sometimes it is necessary to multiply by 1. E.g.: $\int \ln x \; dx = \int \ln x \cdot 1 \; dx \implies f(x) = \ln x,\ g'(x) = 1$.
- Sometimes it is necessary to do it multiple times

**Substitution**

Let $a < b$, $\phi : [a,b] \to \mathbb{R}$, cont. diff, $I \subseteq \mathbb{R}$ with $\phi([a,b]) \subseteq I$ and $f : I \to \mathbb{R}$ a cont. function. Then it follows:

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\phi'(t)dt = (F \circ \phi)(b) - (F \circ \phi)(a)$$

since $F' = f$ then $f(\phi(t))\phi'(t) = (F \circ \phi)'(t)$.

**Partial Fraction Decomposition**

Let $P(x)$, $Q(x)$ be two polynomials. $\int \frac{P(x)}{Q(x)}$ can be calculated as follows:

1. If $\deg(P) \geq Q(P) \Rightarrow$ poly. div. $\frac{P(x)}{Q(x)} = a(x) + \frac{r(x)}{Q(x)}$
2. Calculate all roots of $Q(x)$
3. Create a partial fraction per root
   - Simple real root: $x_1 \to \frac{A}{x-x_1}$
   - $n$-fold real root: $x_1 \to \frac{A_1}{x-x_1} + \ldots + \frac{A_r}{(x-x_1)^r}$
   - Simple $i$-root: $x^2 + px + q \to \frac{Ax+B}{x^2+px+q}$
   - $n$-fold $i$-root: $x^2 + px + q \to \frac{A_1x+B_1}{x^2+px+q} + \ldots + \frac{A_rx+B_r}{(x^2+px+q)^r}$
4. Calculate parameters $A_1, \ldots, A_n$. (Insert the root as $s$, transform and solve)

# Trigonometry

## Periodicity

$\sin(x) = \sin(x + 2\pi)$ $\qquad$ $\cos(x) = \cos(x + 2\pi)$
$\tan(x) = \tan(x + \pi)$ $\qquad$ $\cot(x) = \cot(x + \pi)$

## Parity

$\sin(-x) = -\sin(x)$ $\qquad$ $\cos(-x) = \cos(x)$
$\tan(-x) = -\tan(x)$ $\qquad$ $\cot(-x) = -\cot(x)$

## Complement

$\sin(\pi - x) = \sin(x)$ $\qquad$ $\cos(\pi - x) = -\cos(x)$
$\tan(\pi - x) = -\tan(x)$ $\qquad$ $\cot(\pi - x) = -\cot(x)$

## Multiple-angles formulae

$\sin(2x) = 2 \sin x \cos x$ $\qquad$ $\cos(2x) = \cos^2 x - \sin^2 x$
$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$ $\qquad$ $\cot(2x) = \frac{\cot x - \tan x}{2}$
$\sin(3x) = 3 \sin x - 4 \sin^3 x$ $\qquad$ $\cos(3x) = 4 \cos^3 x - 3 \cos x$

## Addition Theorems

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$
$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$
$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$

## Multiplication

$\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$
$\cos x \cos y = \frac{1}{2}(\cos(x - y) + \cos(x + y))$
$\sin x \cos y = \frac{1}{2}(\sin(x - y) + \sin(x + y))$

## Powers

$\sin^2 x = \frac{1 - \cos(2x)}{2}$ $\qquad$ $\sin^3 x = \frac{3 \sin x - \sin(3x)}{4}$
$\cos^2 x = \frac{1 + \cos(2x)}{2}$ $\qquad$ $\cos^3 x = \frac{3 \cos x + \cos(3x)}{4}$
$\tan^2 x = \frac{1 - \cos(2x)}{1 + \cos(2x)}$

## Sum of functions

$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$
$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$
$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$
$\cos x - \cos y = 2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$

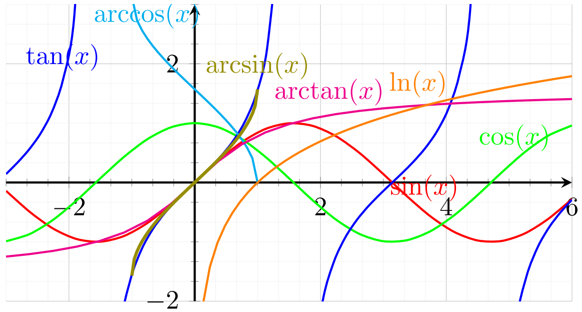
## Miscellaneous

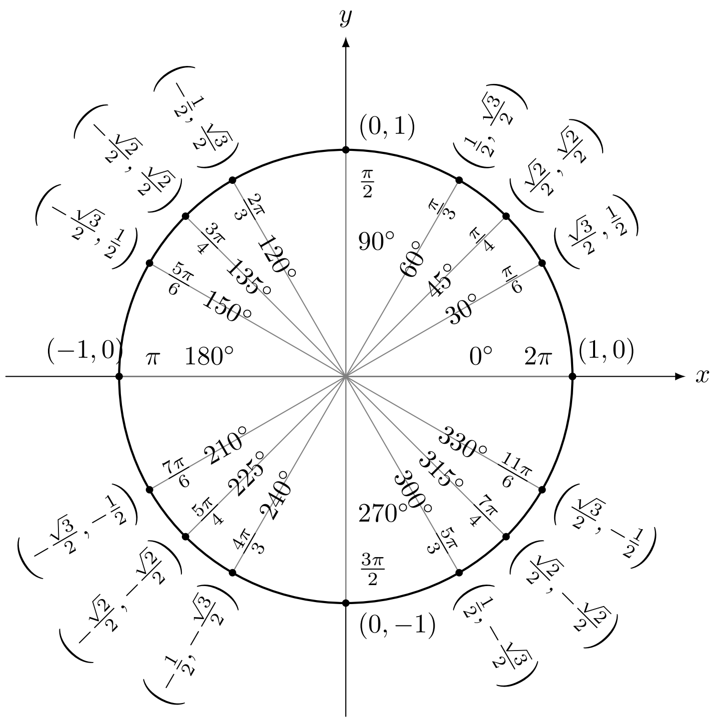
$\sin^2 x + \cos^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$
$\sin x^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$ $\qquad$ $\cos x^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$

## Angles

| deg | 0 | 30 | 45 | 60 | 90 | 120 | 135 | 150 | 180 | 270 | 360 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| rad | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
| sin | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | -1 | 0 |
| cos | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{\sqrt{2}}$ | $-\frac{\sqrt{3}}{2}$ | -1 | 0 | 1 |
| tan | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | - | $-\sqrt{3}$ | $-1$ | $-\frac{1}{\sqrt{3}}$ | 0 | - | 0 |

## Important Functions




## Series

- Geometric: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ if $|q| < 1$
- Harmonic: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges
- Telescope: $\sum_{n=0}^{\infty} \frac{1}{n(n+1)} = 1$
- $\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n\to\infty}(1 + \frac{z}{n})^n = e^z$
- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges $s > 1$ $\left(\frac{1}{1 - \frac{1}{2^{s-1}}}\right)$
- $p(z) = \sum_{k=0}^{\infty} c_k z^k$ conv. abs. $|z| < \rho = \frac{1}{\limsup |c_k|^{1/k}}$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ $\qquad$ $\sum_{i=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

## Taylor Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5)$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \mathcal{O}(x^7)$$
$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \mathcal{O}(x^6)$$
$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6)$$
$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)$$
$$\tanh(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \mathcal{O}(x^7)$$
$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)$$
$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \mathcal{O}(x^4)$$

## Parity of Functions

**Even:** $f(-x) = f(x) \quad \forall x \in D$ $\qquad$ $|x|, \ \cos x, \ x^2$
**Odd:** $f(-x) = -f(x) \quad \forall x \in D$ $\qquad$ $x, \sin, \tan, x^3$

**Chaining of odd functions**
Chaining odd functions results in an odd function.

## Derivatives and Integrals (src: dcamenisch)

| $\mathbf{F(x)}$ | $\mathbf{f(x)}$ |
|---|---|
| $c$ | $0$ |
| $x^a$ | $a \cdot x^{a-1}$ |
| $\frac{1}{a+1}x^{a+1}$ | $x^a$ |
| $\frac{1}{a\cdot(n+1)}(ax+b)^{n+1}$ | $(ax+b)^n$ |
| $\frac{x^{a+1}}{a+1}$ | $x^a,\ a \neq -1$ |
| $\frac{1}{x}$ | $-\frac{1}{x^2}$ |
| $\sqrt{x}$ | $\frac{1}{2\sqrt{x}}$ |
| $\sqrt[n]{x}$ | $\frac{1}{n}x^{\frac{1}{n}-1}$ |
| $\frac{2}{3}x^{\frac{3}{2}}$ | $\sqrt{x}$ |
| $\frac{n}{n+1}x^{\frac{1}{n}+1}$ | $\sqrt[n]{x}$ |
| $e^x$ | $e^x$ |
| $\ln(\lvert x\rvert)$ | $\frac{1}{x}$ |
| $\log_a(\lvert x\rvert)$ | $\frac{1}{x\ln(a)} = \log_a(e^{\frac{1}{x}})$ |
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |
| $\tan(x) = \frac{\sin(x)}{\cos(x)}$ | $\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$ |
| $\cot(x) = \frac{\cos(x)}{\sin(x)}$ | $\frac{1}{-\sin^2(x)}$ |
| $\arcsin(x)$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\arccos(x)$ | $\frac{-1}{\sqrt{1-x^2}}$ |
| $\arctan(x)$ | $\frac{1}{1+x^2}$ |
| $\sinh(x) = \frac{e^x+e^{-x}}{2}$ | $\cosh(x)$ |
| $\cosh(x) = \frac{e^x-e^{-x}}{2}$ | $\sinh(x)$ |
| $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ | $\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$ |
| $\frac{1}{f(x)}$ | $\frac{-f'(x)}{(f(x))^2}$ |
| $a^{cx}$ | $a^{cx} \cdot c\ln(a)$ |
| $x^x$ | $x^x \cdot (1+\ln(x)),\ x>0$ |
| $(x^x)^x$ | $(x^x)^x(x+2x\ln(x)),\ x>0$ |
| $x^{x^x}$ | $x^{x^x}(x^{x-1}+\ln(x)\cdot x^x(1+\ln(x)))$ |

| $\mathbf{F(x)}$ | $\mathbf{f(x)}$ |
|---|---|
| $\frac{1}{a}\ln(\lvert ax+b\rvert)$ | $\frac{1}{ax+b}$ |
| $\frac{ax}{c} - \frac{ad-bc}{c^2}\ln(\lvert cx+d\rvert)$ | $\frac{ax+b}{cx+d}$ |
| $\frac{1}{2a}\ln\left(\left\lvert\frac{x-a}{x+a}\right\rvert\right)$ | $\frac{1}{x^2-a^2}$ |
| $\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)$ | $\sqrt{a^2+x^2}$ |
| $\frac{x}{2}\sqrt{a^2-x^2} - \frac{a^2}{2}\arcsin\left(\frac{x}{\lvert a\rvert}\right)$ | $\sqrt{a^2-x^2}$ |
| $\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2-a^2}\right)$ | $\sqrt{x^2-a^2}$ |
| $\ln\left(x+\sqrt{x^2\pm a^2}\right)$ | $\frac{1}{\sqrt{x^2\pm a^2}}$ |
| $\arcsin\left(\frac{x}{\lvert a\rvert}\right)$ | $\frac{1}{\sqrt{a^2-x^2}}$ |
| $\frac{1}{a}\arctan\left(\frac{x}{a}\right)$ | $\frac{1}{x^2+a^2}$ |
| $-\frac{1}{a}\cos(ax+b)$ | $\sin(ax+b)$ |
| $\cos(ax+b)$ | $-a\sin(ax+b)$ |
| $\frac{1}{a}\sin(ax+b)$ | $\cos(ax+b)$ |
| $\sin(ax+b)$ | $a\cos(ax+b)$ |
| $-\ln(\lvert\cos(x)\rvert)$ | $\tan(x)$ |
| $\ln(\lvert\sin(x)\rvert)$ | $\cot(x)$ |
| $\ln\left(\left\lvert\tan\left(\frac{x}{2}\right)\right\rvert\right)$ | $\frac{1}{\sin(x)}$ |
| $\ln\left(\left\lvert\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\rvert\right)$ | $\frac{1}{\cos(x)}$ |
| $\frac{1}{2}(x-\sin(x)\cos(x))$ | $\sin^2(x)$ |
| $\frac{1}{2}(x+\sin(x)\cos(x))$ | $\cos^2(x)$ |
| $\frac{1}{4}(\frac{1}{3}\cos(3x)-3\cos(x))$ | $\sin^3(x)$ |
| $\frac{1}{4}(\frac{1}{3}\sin(3x)+3\sin(x))$ | $\cos^3(x)$ |
| $\tan(x)-x$ | $\tan^2(x)$ |
| $-\cot(x)-x$ | $\cot^2(x)$ |
| $x\arcsin(x)+\sqrt{1-x^2}$ | $\arcsin(x)$ |
| $x\arccos(x)-\sqrt{1-x^2}$ | $\arccos(x)$ |
| $x\arctan(x)-\frac{1}{2}\ln\left(1+x^2\right)$ | $\arctan(x)$ |
| $\ln(\cosh(x))$ | $\tanh(x)$ |
| $\ln(\lvert f(x)\rvert)$ | $\frac{f'(x)}{f(x)}$ |

| $\mathbf{F(x)}$ | $\mathbf{f(x)}$ |
|---|---|
| $x(\ln(\lvert x\rvert)-1)$ | $\ln(\lvert x\rvert)$ |
| $\frac{1}{n+1}(\ln x)^{n+1} \quad n \neq -1$ | $\frac{1}{x}(\ln x)^n$ |
| $\frac{1}{2n}(\ln x^n)^2 \quad n \neq 0$ | $\frac{1}{x}\ln x^n$ |
| $\ln(\lvert\ln(x)\rvert) \quad x>0, x\neq 1$ | $\frac{1}{x\ln(x)}$ |
| $\frac{1}{b\ln(a)}a^{bx}$ | $a^{bx}$ |
| $\frac{cx-1}{c^2}\cdot e^{cx}$ | $x\cdot e^{cx}$ |
| $\frac{1}{c}e^{cx}$ | $e^{cx}$ |
| $\frac{x^{n+1}}{n+1}\left(\ln(x)-\frac{1}{n+1}\right) \quad n\neq -1$ | $x^n\ln(x)$ |
| $\frac{e^{cx}(c\sin(ax+b)-a\cos(ax+b))}{a^2+c^2}$ | $e^{cx}\sin(ax+b)$ |
| $\frac{e^{cx}(c\cos(ax+b)+a\sin(ax+b))}{a^2+c^2}$ | $e^{cx}\cos(ax+b)$ |
| $\sin(x)\cos(x)$ | $\frac{\sin^2(x)}{2}$ |
| $\frac{1}{2}(f(x))^2$ | $f'(x)f(x)$ |
| $\sqrt{\pi}$ | $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ |
| $\frac{1}{a(n+1)}(ax+b)^{n+1}$ | $(ax+b)^n$ |
| $\frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}$ | $x(ax)^n$ |
| $\frac{(ax^p+b)^{n+1}}{ap(n+1)}$ | $(ax^p+b)^n x^{p-1}$ |
| $\frac{1}{ap}\ln\lvert ax^p+b\rvert$ | $(ax^p+b)^{-1}x^{p-1}$ |
| $\frac{ax}{c} - \frac{ad-bc}{c^2}\ln\lvert cx+d\rvert$ | $\frac{ax+b}{cx+d}$ |
| $-x\cos(x)+\sin(x)$ | $x\sin(x)$ |
| $x\sin(x)+\cos(x)$ | $x\cos(x)$ |
| $\mathrm{arccot}(x)$ | $-\frac{1}{1+x^2}$ |
| $\coth(x)$ | $1-\coth^2 x = -\frac{1}{\sinh^2(x)}$ |
| $\mathrm{arcoth}(x)$ | $\frac{1}{1-x^2}$ |