Analysis I Summary

Nicola Studer
nicstuder@student.ethz.ch

August 5, 2022

1 Real numbers, euclidean spaces

Archimedes' principle. If $x \in \mathbb{R}$ with $x > 0$ and $y \in \mathbb{R}$, then $\exists n \in \mathbb{N}$ $(y \leq n \cdot x)$

Thm. (i) $|x| \geq 0 \quad \forall x \in \mathbb{R}$

(ii) $|xy| = |x||y| \quad \forall x, y \in \mathbb{R}$

(iii) $|x+y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$

(iv) $|x+y| \geq ||x| - |y|| \quad \forall x, y \in \mathbb{R}$

Young's inequality. $\forall \epsilon > 0, \forall x, y \in \mathbb{R}$:

$$2|xy| \leq \epsilon x^2 + \frac{1}{\epsilon} y^2$$

2 Sequences

2.1 Convergence

$(a_n)_{n\geq 1}$ converges to $L = \lim_{n\to\infty} a_n$
$\iff \forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n > N \; (|a_n - L| < \epsilon)$

Def (Convergence). $(a_n)_{n\geq 1}$ converges
$\iff \exists L \in \mathbb{R} \; \forall \epsilon > 0 \; (\{n \in \mathbb{N} \mid |a_n - L| \geq \epsilon\})$ is finite.

Hint. Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ converge with limit a and b:

1. $(a_n + b_n)_{n\geq 1}$ converges with limit $a + b$
2. $(a_n \cdot b_n)_{n\geq 1}$ converges with limit $a \cdot b$.
3. $(\frac{a_n}{b_n})_{n\geq 1}$ converges with limit $\frac{a}{b}$
4. $\exists K \geq 1 \; \forall n \geq K : a_n \leq b_n \implies a \leq b$

2.1.1 Tips & Tricks

- a_n convergent $\implies$ a_n bounded
- a_n convergent $\iff$ a_n bounded and $\liminf a_n = \limsup a_n$

Monotone Convergence. $(a_n)_{n\geq 1}$ monotone increasing and upper bounded $\implies \lim a_n = \sup\{a_n \mid n \geq 1\}$
$(a_n)_{n\geq 1}$ monotone decreasing and lower bounded $\implies \lim a_n = \inf\{a_n \mid n \geq 1\}$

Lemma (Bernoulli Inequation).

$$(1+x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}, x > -1$$

Def (Limit inferior / Limit superior).

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} (\inf\{a_k \mid k \geq n\})$$

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} (\sup\{a_k \mid k \geq n\})$$

Cauchy Criteria. a_n converges iff $\forall \epsilon > 0 \; \exists N \geq 1$ s.t. $|a_n - a_m| < \epsilon \; \forall n, m \geq N$ (cauchy sequence).

(i) Each Cauchy sequence is bounded

(ii) $(a_n)_{n\geq 1}$ conv. $\implies (a_n)_{n\geq 1}$ cauchy

(iii) $(a_n)_{n\geq 1}$ cauchy $\implies (a_n)_{n\geq 1}$ conv.

Bolzano-Weierstrass. Each bounded sequence contains a convergent sub sequence.

Sandwich. If $\lim a_n = \alpha$, $\lim b_n = \alpha$, $k \in \mathbb{N}$ and $a_n \leq c_n \leq b_n \; \forall n \geq k$, then $\lim c_n = \alpha$

Cauchy-Cantor. Let $I_1 \supseteq I_2 \supseteq \cdots I_n \cdots$ be a sequence of proper intervals with $\mathcal{L} < +\infty$, then $\bigcap_{n\geq 1} I_n \neq 0$. And if $\lim_{n\to\infty} \mathcal{L}(I_n) = 0$ then $|\bigcap_{n\geq 1} I_n| = 1$

Cor. *Let (a_n) be bounded, then for each subsequence (b_n): $\liminf a_n \leq \lim b_n \leq \limsup a_n$.*

Each subsequence (b_n) of a convergent (a_n) converges and $\lim b_n = \lim a_n$.

3 Series

Def. "$\sum_{k=1}^\infty a_k$" converges, if the sequence $(S_n)_{n\geq 1}$ of partial sums converges and $\sum\limits_{k=1}^\infty a_k := \lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^n a_k$

Thm. $\sum_{k=1}^\infty a_k$ and $\sum_{j=1}^\infty b_j$ convergent:

- $\sum_{k=1}^\infty (a_k + b_k) = (\sum_{k=1}^\infty a_k) + (\sum_{k=1}^\infty b_k)$
- $\sum_{k=1}^\infty \alpha \cdot a_k = \alpha \sum_{k=1}^\infty a_k$

Couchy Criteria. $\sum_{k=1}^\infty a_k$ conv. $\iff \forall \epsilon > 0 \; \exists N \geq 1 :$ $|\sum_{k=n}^m a_k| = |S_m - S_n| < \epsilon \; \forall m \geq n \geq N$

Zero Sequence Criteria. $\sum\limits_{k=1}^\infty a_k$ conv. $\implies \lim a_k = 0$

Comparison Theorem. Let $\sum_{k=1}^\infty a_k$, $\sum_{k=1}^\infty b_k$ series s.t. $0 \leq a_k \leq |a_k| \leq b_k \quad \forall k \geq 1$:

$$\sum_{k=1}^\infty b_k \text{ converges} \implies \sum_{k=1}^\infty a_k \text{ converges absolute}$$

$$\sum_{k=1}^\infty a_k \text{ diverges} \implies \sum_{k=1}^\infty b_k \text{ diverges}$$

Thm. Let $\sum_{k=1}^\infty a_k$ be a series with $a_k \geq 0 \quad \forall k \in \mathbb{N}^*$

$$\sum_{k=1}^\infty \text{ converges} \iff (S_n)_{n\geq 1} \text{ upper bounded}$$

Def (Asolute Convergence). $\sum_{k=1}^\infty a_k$ absolute converges if $\sum_{k=1}^\infty |a_k|$ converges.

$$\sum_{k=1}^\infty |a_k| \text{ converges} \implies \sum_{k=1}^\infty a_k \text{ converges}$$

$$\sum_{k=1}^\infty a_k \text{ converges} \not\Rightarrow \sum_{k=1}^\infty |a_k| \text{ converges}$$

(**Dirichlet**) If a series converges absolute, then each permutation of the series converges with the same limit.
(**Riemann**) If a series only converges, then there exists a permutation such that:

$$\sum_{k=1}^\infty a_{\phi(k)} = x \quad \forall x \in \mathbb{R} \cup \{\infty\}$$

Thm.

$$\left|\sum_{k=1}^\infty a_k\right| \leq \sum_{k=1}^\infty |a_k|$$

Leibniz. (a_n) mon. dec. s.t. $a_n \geq 0 \, \forall n \geq 1 \wedge \lim a_n = 0$:

$$S := \sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ converges.}$$

Furthermore: $a_1 - a_2 \leq S \leq a_1$

Ratio Test. Let $(a_n)_{n\geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:

$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute.

$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n$ diverges

Lemma. $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n$ *converges absolute.*
- $L > 1 \implies \sum_{n=1}^{\infty} a_n$ *diverges.*
- $L = 1 \implies$ *no information*

Root Test. Let $(a_n)_{n\geq 1}$ with $a_n \neq 0 \quad \forall n \geq 1$:

$\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1 \implies \sum_{n=1}^{\infty} a_n$ converges absolute.

$\liminf_{n\to\infty} \sqrt[n]{|a_n|} > 1 \implies \sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} |a_n|$ diverge.

Lemma. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$:

- $L < 1 \implies \sum_{n=1}^{\infty} a_n$ *converges absolute.*
- $L > 1 \implies \sum_{n=1}^{\infty} a_n$ *diverges.*
- $L = 1 \implies$ *no information*

Def (Cauchy Product)**.** $\sum_{i=0}^{\infty} a_i$, $\sum_{j=0}^{\infty} b_j$:

$\sum_{n=0}^{\infty} (\sum_{j=0}^{n} a_{n-j} b_j) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$

Thm. $\sum_{i=0}^{\infty} a_i$, $\sum_{j=0}^{\infty} b_j$ conv. abs. $\Rightarrow$ Couchy prod. conv. abs.:

$$\sum_{n=0}^{\infty} (\sum_{j=0}^{n} a_{n-j} b_j) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_i b_j = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} a_i b_j$$

Hint (Strategy: Convergence of Series)**.**

1. Check for known types. Telescope, Geometric, etc.
2. $\lim |a_n| \neq 0 \implies$ divergence
3. Ratio Test
4. Root Test
5. Search convergent majors: $0 \leq a_n \leq b_n$
6. If divergent minors $\implies$ divergence
7. Be creative

4 Functions

$\mathbb{R}^D = \{f : D \to \mathbb{R} \mid f \text{ is function}\}$, $(\mathbb{R}^D; +, \cdot)$ is V.R.

4.1 Continuity

Def (Continuity)**.** A function f is continuous in x_0 if:

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall x \in D \, (|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon)$$

$$\iff$$

$$\forall (a_n) \text{ with } \lim_{n\to\infty} a_n = x_0 : \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(x_0)$$

Def. A function $f : D \to \mathbb{R}$ is continuous if it is continuous in all $x_0 \in D$

Hint. To prove continuity try to filter $|x - x_0|$ out of $|f(x) - f(x_0)|$ and choose δ, such that the rest term disappears. Be aware that δ is part of ϵ and normally $|x_0|$ as well. But not x!

Cor. *$f, g : D \to \mathbb{R}$ continuous in $x_0 \in D$. Then:*

- *$fg, \lambda f, f \pm g$ continuous in x_0*
- *$\frac{f}{g} : D \setminus \{x \in D \mid g(x) = 0\} \to \mathbb{R}$ continuous in x_0 (if $g(x_0) \neq 0$)*
- *$|f|, \max(f, g), \min(f, g)$ continuous in x_0*
- *$P(x) = a_n x^n + \cdots + a_0$ continuous on $\mathbb{R}$*
- *$\frac{P(x)}{Q(x)}$ continuous on $\mathbb{R} \setminus \{x_1, \cdots, x_m\}$ if $x_1, \cdots, x_m$ are roots of $Q(x)$*

Thm. Let $f : D_1 \to D2 \subset \mathbb{R}, g : D_2 \to \mathbb{R}$ be continuous $\implies g \circ f : D_1 \to \mathbb{R}$ continuous.

Bolzano (Intermediate value theorem). Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ continuous and $a, b \in I$. For each c between $f(a)$ and $f(b)$ there is a $z \in [a, b]$ with $f(z) = c$

Min-Max. Let $f : I = [a, b] \to \mathbb{R}$ be continuous.

$$\exists u, v \in I \; \forall x \in I \; (f(u) \leq f(x) \leq f(v))$$

In particular $f([a, b]) \subset [f(u), f(v)]$ is bounded.

Cor. *$I = [a, b]$, $f : I \to \mathbb{R}$ continuous, then $Im(f) = f(I)$ is a compact interval $J = [\min f, \max f] = [f(u), f(v)]$*

Inverse Mapping. Let $f : I \to \mathbb{R}$ be continuous and strict monotone increasing. Then $J := f(I) \subseteq \mathbb{R}$ is an interval and $f^{-1} : J \to I$ is continuous and strict monotone.

4.2 Exponential function

$\exp : \mathbb{R} \to]0, \infty[$ is continuous, strictly monotone increasing, surjective.

- $\exp(x) \geq 1 + x \quad \forall x \in \mathbb{R}$
- For $x > 0, a \in \mathbb{R} : x^a := \exp(a \ln x)$
- $x^0 = 1 \quad \forall x > 0$

Def. The inverse mapping of $\exp(x)$ is called the natural logarithm:

$$\ln :]0, \infty[\to \mathbb{R}, \quad x \mapsto \ln x$$

It is strictly monotone increasing, continuous and bijective.

4.3 Converge of function sequences

$$\mathbb{N} \to \mathbb{R}^D = \{f : D \to \mathbb{R}\}, \quad n \mapsto f(n)$$

Def (pointwise convergence)**.** $(f_n)_{n\geq 0}$ converges pointwise to a function $f : D \to \mathbb{R}$, if $\forall x \in D : \lim_{n\to\infty} f_n(x) = f(x)$

$$\iff$$

$$\forall x \in D \; \forall \epsilon > 0 \; \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \; (|f_n(x) - f(x)| < \epsilon)$$

Def (uniform convergence (Weierstrass))**.** $f_n : D \to \mathbb{R}$ converges uniformly in D to $f : D \to \mathbb{R}$ if:

$$\forall \epsilon > 0 \; \exists N \geq 1 \text{ s.t. } \forall n \geq N \; \forall x \in D \; (|f_n(x) - f(x)| < \epsilon)$$

$$\iff$$

$$\lim_{n\to\infty} \sup_{x\in D} |f_n(x) - f(x)| = 0$$

The function sequence (f_n) is uniformly convergent if for all $x \in D$ the limit $\lim_{n\to\infty} f_n(x) = f(x)$ exists and the sequence (f_n) uniformly converges to f. Furthermore if $\forall \epsilon > 0 \; \exists N \geq 1 \; \forall n, m \geq N \; \forall x \in D : |f_n(x) - f_m(x)| < \epsilon$.

The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly (in D), if the function sequence $S_n(x) := \sum_{k=0}^{n} f_k(x)$ converges uniformly.

Thm. let $D \subseteq \mathbb{R}$ and $f_n : D \to \mathbb{R}$ a function sequence containing (in D) continuous functions which converge (in D) uniformly against a function $f : D \to \mathbb{R}$, then f (in D) is continuous.

Hint (not uniform convergent). $(f_n)_{n\geq 0}$ converges not uniformly if: $\forall \epsilon > 0\ \forall N \in \mathbb{N}\ \exists x \in D(|f_n(x) - f(x)| \geq \epsilon)$

Hint. Check the function and try to construct x (dependent on N in general), such that $|f_n(x) - f(x)|$ is always greater than a specific ϵ and afterwards choose the ϵ.

Def (Power Functions). $\sum_{k=0}^{\infty} c_k x^k$ has positive convergence radius if $\limsup_{k\to\infty} \sqrt[k]{|c_k|}$ exists.

$$\rho = \begin{cases} +\infty & , \text{if } \limsup_{k\to\infty} \sqrt[k]{|c_k|} = 0 \\ \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|c_k|}} & , \text{if } \limsup_{k\to\infty} \sqrt[k]{|c_k|} > 0 \end{cases}$$

Thm. Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $\rho > 0$ and let $f(x) = \sum_{k=0}^{\infty} c_k x^k, |x| < \rho$ Then: $\forall 0 \leq r < \rho$ converges $\sum_{k=0}^{\infty} c_k x^k$ uniformly on $[-r, r]$, furthermore $f :]-\rho, \rho[\to \mathbb{R}$ is continuous.

4.4 Trigonometric Functions

sin and cos are continuous functions $\mathbb{R} \to \mathbb{R}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Thm.
1. $\exp(iz) = \cos(z) + i\sin(z) \quad \forall z \in \mathbb{C}$
2. $\cos z = \cos(-z)$ und $\sin(-z) = -\sin(z) \quad \forall z \in \mathbb{C}$
3. $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$
4. $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$
   $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
5. $\cos(z)^2 + \sin(z)^2 = 1 \quad \forall z \in \mathbb{C}$

Cor.

$$\sin(2z) = 2\sin(z)\cos(z)$$

$$\cos(2z) = \cos(z)^2 - \sin(z)^2$$

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

Def (π). $\pi := \inf\{t > 0 \mid \sin t = 0\}$

(i) $\sin \pi = 0, \pi \in]2, 4[$

(ii) $\forall x \in]0, \pi[: \sin x > 0$

(iii) $e^{\frac{i\pi}{2}} = i$

Cor. $x \geq \sin x \geq x - \frac{x^3}{3!} \quad \forall 0 \leq x \leq \sqrt{6}$

Cor.
1. *$e^{i\pi} = -1, \qquad e^{2i\pi} = 1$*
2. $\sin(x + \frac{\pi}{2}) = \cos(x), \quad \cos(x + \frac{\pi}{2}) = -\sin(x) \quad \forall x \in \mathbb{R}$
3. $\sin(x + \pi) = -\sin(x), \quad \sin(x + 2\pi) = \sin(x) \quad \forall x \in \mathbb{R}$
4. $\cos(x + \pi) = -\cos(x), \quad \cos(x + 2\pi) = \cos(x) \quad \forall x \in \mathbb{R}$
5. *Roots of sinus* $= \{\pi \cdot k \mid k \in \mathbb{Z}\}$
   $\sin(x) > 0\ \forall x \in]2k\pi, (2k+1)\pi[, \quad k \in Z$
   $\sin(x) < 0\ \forall x \in](2k+1)\pi, (2k+2)\pi[, \quad k \in \mathbb{Z}$
6. *Roots of cosine* $= \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$
   $\cos(x) > 0\ \forall x \in]-\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} + (2k+1)\pi[, \quad k \in \mathbb{Z}$
   $\cos(x) < 0\ \forall x \in]-\frac{\pi}{2} + (2k+1)\pi, -\frac{\pi}{2} + (2k+2)\pi[, \quad k \in \mathbb{Z}$

4.5 Limit of Functions

Def (accumulation point). $x_0 \in \mathbb{R}$ is an accumulation point of D if $\forall \delta > 0$: $(]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) \cap D \neq \varnothing$

Def (Limit of Function). if $f : D \to \mathbb{R}, x_0 \in \mathbb{R}$ an accumulation point of D, then $A \in \mathbb{R}$ is the limit of $f(x)$ for $x \to x_0$, written as $\lim_{x\to x_0} f(x) = A$. If $\forall \epsilon > 0\ \exists \delta > 0$ s.t. $\forall x \in D \cap (]x_0 - \delta, x_0 + \delta[\setminus \{x_0\}) : \ |f(x) - A| < \epsilon$

Important Rules. Let $f : D \to \mathbb{R}$ and x_0 is an accumulation point of D.

1. $\lim_{x\to x_0} f(x) = A \iff \forall (a_n)_{n\geq 1}$ in $D \setminus \{x_0\}$ with $\lim_{n\to\infty} a_n = x_0 \implies \lim_{n\to\infty} f(a_n) = A$.
2. Let $x_0 \in D$. Then f is continuous in x_0 $\iff \lim_{x\to x_0} f(x) = f(x_0)$
3. $f, g : D \to \mathbb{R}$ and $\exists \lim_{x\to x_0} f(x), \exists \lim_{x\to x_0} g(x) \implies$
   $$\lim_{x\to x_0} (f+g)(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$
   $$\lim_{x\to x_0} (f \cdot g)(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$$
4. $f, g : D \to \mathbb{R}$ and $f \leq g$, then if both limit exists
   $$\lim_{x\to x_0} f(x) \leq \lim_{x\to x_0} g(x)$$
5. If $g_1 \leq f \leq g_2$ and $\lim_{x\to x_0} g_1(x) = \lim_{x\to x_0} g_2(x)$ then $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g_1(x)$

Hint. Sometimes it can be really helpful to convert known functions to their power series to calculate a limit. E.g.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \dots}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3!} + \dots = 1$$

Hint ($e^{\log}$). Transform ugly function with this trick.

$$\lim_{x\to x_0} f(x)^{g(x)} = \lim_{x\to x_0} e^{g(x)\log(f(x))} = e^{\lim_{x\to x_0} g(x)\log(f(x))}$$

5 Differentiable Functions

Def (Differentiable). f is in x_0 differentiable, if the limit $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = f'(x_0)$ exists. f is differentiable if $\forall x_0 \in D$ f is differentiable.

Weierstrass. $f : D \to \mathbb{R}$, $x_0 \in D$ accumulation point. Equivalent statements:

1. f is in x_0 differentiable
2. It exists $c \in R$ $(c = f'(x_0))$ and $r : D \to \mathbb{R}$ s.t.:
   2.1 $f(x) = f(x_0) + c(x - x_0) + r(x)(x - x_0)$
   2.2 $r(x_0) = 0$ and r continuous in x_0.

Cor. *f diff. in $x_0 \implies f$ continuous in x_0*

Thm. f diff. in $x_0 \iff \exists \phi : D \to \mathbb{R}$ continuous in x_0 s.t. $\forall x \in D : f(x) = f(x_0) + \phi(x)(x - x_0)$. Then $\phi(x_0) = f'(x_0)$.

Derivative rules.
Linearity: $(\alpha \cdot f(x) + g(x))' = \alpha \cdot f'(x) + g'(x)$
Product rule: $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g(x)^2}$
Chain rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Cor. *f bijective and in x_0 differentiable s.t. $f'(x_0) \neq 0$. f^{-1} is continuous in $y_0 = f(x_0)$. Then f^{-1} is differentiable in y_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$.*

5.1 Derivative Implications

1. x_0 is local minimum if $f'(x_0) = 0 \wedge f''(x_0) > 0$ or the sign of f' changes from $-$ to $+$.
2. x_0 is local maximum if $f'(x_0) = 0 \wedge f''(x_0) < 0$ or the sign of f' changes from $+$ to $-$.
3. x_0 is local extremum if $f'(x_0) = 0 \wedge f''(x_0) \neq 0$
4. x_0 is a saddle point if $f'(x_0) = 0$ and $f''(x_0) = 0$
5. x_0 is a inflection point if $f''(x_0) = 0 \wedge f^{(3)}(x_0) \neq 0$
6. $f'(x_0) = f^{(2)}(x_0) = \ldots = f^{(n)}(x_0) = 0$
   6.1 n odd and $f^{(n+1)}(x_0) > 0 \implies x_0$ strict local minimum
   6.2 n odd and $f^{(n+1)}(x_0) < 0 \implies x_0$ strict local maximum

5.2 Derivative Theorems

Rolle. Let $f : [a,b] \to \mathbb{R}$ continuous and in $]a,b[$ differentiable. If $f(a) = f(b)$, then there exists $\xi \in]a,b[$ with $f'(\xi) = 0$.

Mean Value / Lagrange. Let $f : [a,b] \to \mathbb{R}$ continuous and in $]a,b[$ differentiable, then there exists $\xi \in]a,b[$ with $f(b) - f(a) = f'(\xi)(b-a)$.
There exists points ξ with $f'(\xi)$ equal to the gradient of the secant between a to b.

Cor. *Let $f, g : [a,b] \to \mathbb{R}$ cont. and diff. in $]a,b[$.*

1. *$\forall \xi \in]a,b[: f'(\xi) = 0 \implies f$ is constant*
2. *$\forall \xi \in]a,b[: f'(\xi) = g'(x) \implies \exists c \in \mathbb{R} \forall x \in [a,b] : f(x) = g(x) + c$*
3. *$\forall \xi \in]a,b[: f'(\xi) \geq 0 \implies f$ in $[a,b]$ mon. inc.*
4. *$\forall \xi \in]a,b[: f'(\xi) > 0 \implies f$ in $[a,b]$ str. mon. inc.*
5. *$\forall \xi \in]a,b[: f'(\xi) \leq 0 \implies f$ in $[a,b]$ mon. dec.*
6. *$\forall \xi \in]a,b[: f'(\xi) < 0 \implies f$ in $[a,b]$ str. mon. dec.*
7. *$\exists M \geq 0 \, \forall \xi \in]a,b[: |f'(\xi)| \leq M \implies \forall x_1, x_2 \in [a,b] : |f(x_1) - f(x_2)| \leq M|x_1 - x_2|$*

Cauchy. $f, g : [a,b] \to \mathbb{R}$ continuous and in $]a,b[$ diff. Then there exists $\xi \in]a,b[$ with
$g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a))$.
If $\forall x \in]a,b[: g'(x) \neq 0$ it implies that $g(a) \neq g(b)$ and $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$

l'Hôspital. $f, g :]a,b[\to \mathbb{R}$ diff. with $\forall x \in]a,b[: g'(x) \neq 0$. If $\lim_{x \to b^-} f(x) = 0$, $\lim_{x \to b^-} g(x) = 0$ and $\lim_{x \to b^-} \frac{f'(x)}{g'(x)} =: \lambda$ exists, then $\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lim_{x \to b^-} \frac{f'(x)}{g'(x)}$.

Hint. Only use l'Hospital if either $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

Def.
1. (strictly) convex: $(x \leq y)$: $f(\lambda x + (1-\lambda)y)$ $(<)$ $\leq \lambda f(x) + (1-\lambda)f(y)$
2. (strictly) concave: $(x \leq y)$: $f(\lambda x + (1-\lambda)y)$ $(>)$ $\geq \lambda f(x) + (1-\lambda)f(y)$

Lemma. *$f : I \to \mathbb{R}$. f is convex $\iff \forall x_0 < x < x_1 \in I : \frac{f(x)-f(x_0)}{x-x_0} \leq \frac{f(x_1)-f(x)}{x_1-x}$. Strictly convex if $<$.*

Lemma. *$f :]a,b[\to \mathbb{R}$ diff.*
- *f' (strictly) mon. inc. dec. $\Rightarrow f$ (strictly) conv. conc.*
- *$f'' \geq (>) \; 0 \implies f$ (strictly) conv. ($<$ / $\leq$ for conc.)*

5.3 Higher Derivatives

1. For $n \geq 2$ is f **n-times differentiable in** D if $f^{(n-1)}$ in D is differentiable. Then $f^{(n)} := (f^{(n-1)})'$ and is the n-th derivative of f
2. f is **n-times continuous differentiable in** D if f is n-times differentiable and if $f^{(n)}$ is continuous in D
3. f is in D **smooth** if $\forall n \geq 1$, f is n-times differentiable.

Smooth Functions. exp, sin, cos, sinh, cosh, tanh, ln, arcsin, arccos, arccot, arctan and all polynomials. tan is smooth on $\mathbb{R} \setminus \{\pi/2 + k\pi\}$ and cot on $\mathbb{R} \setminus \{k\pi\}$

Thm. $f, g : D \to \mathbb{R}$ are n-times diff. in D.

1. $(f+g)^{(n)} = f^{(n)} + g^{(n)}$
2. $(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$
3. $(g \circ f)^{(n)}(x) = \sum_{k=1}^{n} A_{n,k}(x)(g^{(k)} \circ f)(x)$ with $A_{n,k}$ as polynomial in the functions $f', f^{(2)}, \ldots, f^{(n+1-k)}$

5.4 Power Series and Taylor approximation

Thm. Let $f_n :]a,b[$ be a function sequence with f_n one time in $]a,b[$ continuous diff. $\forall n \geq 1$. Assume that $(f_n)_{n \geq 1}$, $(f_n')_{n \geq 1}$ uniformly convergent in $]a,b[$ with $\lim_{n\to\infty} f_n =: f$ and $\lim_{n \to \infty} f_n' =: p$, then f is continuously diff. and $f' = p$.

Thm. Let $\sum_{k=0}^{\infty} c_k x^k$ be a power series with convergent radius $\rho > 0$. Then $f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$ is differentiable on $]x_0 - \rho, x_0 + \rho[$ and $\forall x \in]x_0 - \rho, x_0 + \rho[$:
$f'(x) = \sum_{k=0}^{\infty} k c_k (x - x_0)^{k-1}$

Cor. *$f^{(j)} = \sum_{k=j}^{\infty} c_k \frac{k!}{(k-j)!}(x - x_0)^{k-j}$. Furthermore $c_j = \frac{f^{(j)}(x_0)}{j!}$. Power series can be differentiated part by part in their converge area.*

Def (Taylor Polynomial)**.** The n-th Taylor-polynomial of cont. $n+1$ times diff. in $[c,d]$ f is defined as $T_n(f,x,a)$ with center $a \in]c,d[$ and error $R_n(f,x,a)$.
$\forall x \in [a,b] \, \exists \xi \in]x,a[\cup]a,x[$:

$$T_n(f,x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (x-a)^k$$
$$R_n(f,x,a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$
$$f(x) = T_n(f,x,a) + R_n(f,x,a)$$

Hint. The error can be approximated as
$$|R_n(f,x,a)| \leq \sup_{a<c<x} \left| \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \right|$$

Def (Taylor Series)**.** $T_\infty(f,x,x_0) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$

6 Riemann Integral

$a < b$, $I = [a,b]$, $\mathcal{P}(I) = \{P \mid P \subsetneq I \wedge \{a,b\} \in P \wedge |P| \in \mathbb{N}\}$

Def (Partition).
- Partition: $P \in \mathcal{P}(I)$
- $\delta_i := x_i - x_{i-1}$ length of $I_i := [x_{i-1}, x_i]$, $i \geq 1$
- Mesh of partition: $\delta(P) := \max_{1 \leq i \leq n}(x_i, x_{i-1})$
- $\xi := \{\xi_1, \dots, \xi_n\}$, $\xi_i \in I_i$
- P' refines P if $P \subseteq P'$

Def (Riemann Sums). $S(f,P,\xi) := \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$
- Lower sum: $\underline{S}(f,P) := \sum_{i=1}^{n} (\inf_{x \in I_i} f(x))(x_i - x_{i-1})$
- Upper sum: $\overline{S}(f,P) := \sum_{i=1}^{n} (\sup_{x \in I_i} f(x))(x_i - x_{i-1})$

It holds: $-M(b-a) \leq \underline{S}(f,P) \leq \overline{S}(f,P) \leq M(b-a)$

Lemma. $P \subseteq P'$: $\underline{S}(f,P) \leq \underline{S}(f,P') \leq \overline{S}(f,P') \leq \overline{S}(f,P)$

Lemma. $\forall P_1, P_2 \in \mathcal{P}(I) : \underline{S}(f,P_1) \leq \overline{S}(f,P_2)$

Def (Lower Riemann Integral). $\underline{S}(f) := \sup_{P \in \mathcal{P}(I)} \underline{S}(f,P)$

Def (Upper Riemann Integral). $\overline{S}(f) := \inf_{P \in \mathcal{P}(I)} \overline{S}(f,P)$

6.1 Integrability criteria

Def (Integrable). Bounded $f : [a,b] \to \mathbb{R}$ is integrable if $\underline{S}(f) = \overline{S}(f)$ and the shared value is $\int_a^b f(x)\,dx$.

Riemann Criteria. Bounded $f : I \to \mathbb{R}$ is integrable. Let $\mathcal{P}_\delta(I) := \{P \in \mathcal{P}(I) \mid \delta(P) < \delta\}$.

$\Leftrightarrow \quad \forall \epsilon > 0\ \exists P \in \mathcal{P}(I) : \overline{S}(f,P) - \underline{S}(f,P) < \epsilon$

$\Leftrightarrow \quad \forall \epsilon > 0\ \exists \delta > 0\ \forall P \in \mathcal{P}_\delta(I) : \overline{S}(f,P) - \underline{S}(f,P) < \epsilon$

$\Leftrightarrow \quad \forall \epsilon > 0\ \exists \delta > 0\ \forall P \in \mathcal{P}$ with $\delta(P) < \delta$:

$$\left| A - \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \right| < \epsilon$$

Hint. Bounded $f : [a,b] \to \mathbb{R}$ is int. if $\lim_{\delta(P) \to 0} S(f,P,\xi)$ exists for all P with $\delta(P) \to 0$. It follows that $\lim_{\delta(P)\to 0} S(f,P,\xi) = \int_a^b f(x)\,dx$

6.2 Integrable Functions

1. f (bounded) cont. in $[a,b] \implies f$ int. over $[a,b]$
2. f monotone in $[a,b] \implies f$ int. over $[a,b]$
3. If f,g bounded and int., then integrable as well:
$$f+g, \lambda \cdot f, f \cdot g, |f|, \min(f,g), \max(f,g), \frac{f}{g}$$
4. All polynomials are integrable, even $\frac{P(x)}{Q(x)}$ if $Q(x)$ has no root in $[a,b]$

Hint. Let $V := \{f : I \to \mathbb{R} \mid f \text{ is a mapping}\}$. $(V, +, \cdot)$ is a vector space. Then it implies that $W := \{f : I \to \mathbb{R} \mid f \text{ is integrable}\}$ is a subspace of V.

6.3 Integration Inequalities and Theorems

Def (Uniform continuous).

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x,y \in D : |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Thm. $f : [a,b] \to \mathbb{R}$ cont. $\implies f$ is uni. cont. in $[a,b]$.

Thm. f uni. cont. $\implies f$ cont.

Thm. $f,g : [a,b] \to \mathbb{R}$ bounded and integrable and $\forall x \in [a,b] : f(x) \leq g(x) \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Cauchy-Schwarz.

$$\left| \int_a^b f(x)g(x)\,dx \right| \leq \sqrt{\int_a^b f^2(x)\,dx} \sqrt{\int_a^b g^2(x)\,dx}$$

Hint. $< f,g > := \int_a^b f(x)g(x)\,dx$ is a scalar product. $||f||^2 = < f,f > = \int_a^b f^2(x)\,dx$.

Mean Value Theorem. $f : [a,b] \to \mathbb{R}$ continuous $\implies \exists \xi \in [a,b] : \int_a^b f(x)\,dx = f(\xi)(b-a)$

Cauchy. $f,g : [a,b] \to \mathbb{R}$ with f continuous and g bounded and integrable with $g(x) \geq 0$, $\forall x \in [a,b]$

$$\implies \exists \xi \in [a,b] : \int_a^b f(x)g(x)\,dx = f(\xi) \int_a^b g(x)\,dx$$

6.4 Integration Properties

Additive Property.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Linearity.

$$\int_a^b (\alpha f_1 + \beta f_2)\,dx = \alpha \int_a^b f_1(x)\,dx + \beta \int_a^b f_2(x)\,dx$$

Preservation of Order.

$$\forall x \in [a,b] : f(x) \leq g(x) \implies \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Triangle Inequality.

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

6.5 Primitive Functions

Def (Primitive Function). $F : [a,b] \to \mathbb{R}$ is a primitive function of f if F is cont. diff. and $F' = f$.

Hint. f is integrable $\not\Rightarrow$ exists a primitive function for f.

HID. Let $a < b$, $f : [a,b] \to \mathbb{R}$ continuous. The function

$$F(x) := \int_a^x f(t)dt \quad a \leq x \leq b$$

is cont. diff. in $[a,b]$ and $F'(x) = f(x)\ \forall x \in [a,b]$.

Fundamental theorem of calculus. $f : [a,b] \to \mathbb{R}$ continuous. Then there exists a unique (except a constant term) primitive function F of f, such that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

6.6 Integration Methods

Partial Inegration.

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

$$\int_a^b f(x)g'(x)\,dx = (f\cdot g)|_a^b - \int_a^b f'(x)g(x)\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

- Choose g': exp $\to$ trig $\to$ poly $\to$ inverse trig. $\to$ logs
- Choose f: logs $\to$ inverse trig. $\to$ poly $\to$ trig $\to$ exp
- Sometimes it is necessary to multiply by 1. E.g.: $\int \ln x\ \ dx = \int \ln x \cdot 1\ \ dx \implies f(x) = \ln x,\ g'(x) = 1.$
- Sometimes it is necessary to do it multiple times

Substitution. Let $a < b$, $\phi : [a,b] \to \mathbb{R}$, cont. diff, $I \subseteq \mathbb{R}$ with $\phi([a,b]) \subseteq I$ and $f : I \to \mathbb{R}$ a cont. function. Then it follows:

$$\int_{\phi(a)}^{\phi(b)} f(x)\,dx = \int_a^b f(\phi(t))\phi'(t)dt = (F\circ\phi)(b) - (F\circ\phi)(a)$$

since $F' = f$ then $f(\phi(t))\phi'(t) = (F\circ\phi)'(t)$.

Partial Fraction Decomposition. Let $P(x)$, $Q(x)$ be two polynomials. $\int \frac{P(x)}{Q(x)}$ can be calculated as follows:

1. If $\deg(P) \geq Q(P) \Rightarrow$ poly. div. $\frac{P(x)}{Q(x)} = a(x) + \frac{r(x)}{Q(x)}$
2. Calculate all roots of $Q(x)$
3. Create a partial fraction per root
   - Simple real root: $x_1 \to \frac{A}{x-x_1}$
   - n-fold real root: $x_1 \to \frac{A_1}{x-x_1} + \ldots + \frac{A_r}{(x-x_1)^r}$
   - Simple i-root: $x^2 + px + q \to \frac{Ax+B}{x^2+px+q}$
   - n-fold i-root: $x^2+px+q \to \frac{A_1x+B_1}{x^2+px+q} + \ldots + \frac{A_rx+B_r}{(x^2+px+q)^r}$
4. Calculate parameters $A_1, \ldots, A_n$. (Insert the root as s, transform and solve)

Hint (Odd functions)**.** $\int_{-\lambda}^{\lambda} f(x)\,dx = 0$.

Cor. $\int_{a+c}^{b+c} f(x)\,dx = \int_a^b f(t+c)dt$

Cor. $\int_a^b f(ct)dt = \frac{1}{c}\int_{ac}^{bc} f(x)\,dx$

6.7 Integration of convergent series

Thm. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of bounded, integrable functions which converge uniformly against a function $f : [a,b] \to \mathbb{R}$. Then f is bounded and integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx$$

$$\sum_{n=0}^{\infty} \int_a^b f_n(x)\,dx = \int_a^b \left(\sum_{n=0}^{\infty} f_n(x)\,dx\right)$$

Thm. Let $f(x) := \sum_{k=0}^{\infty} c_k x^k$ be a power series with positive convergence radius $p > 0$. Then $\forall 0 \leq r < p$ f is integrable on $[-r,r]$ and $\forall x \in]-p,p[$: $\int\limits_0^x f(t)dt = \sum\limits_{k=0}^{\infty} c_k \frac{x^{k+1}}{k+1}$

6.8 Improper Integral

$$f : [a,\infty) \to \mathbb{R}, f : [-\infty, a] \to \mathbb{R}, f : (-\infty,\infty) \to \mathbb{R}$$

Def. Let $f : [a,\infty[\to \mathbb{R}$ be bounded and integrable on $[a,b]$ for all $b > a$. If $\lim_{b\to\infty} \int_a^b f(x)\,dx$ exists, the limit is defined as $\int_a^\infty f(x)\,dx$ and one can say that f is integrable on $[a,+\infty[$. If the limit does not exists, one can say that $\int_a^\infty f(x)\,dx$ diverges.

Comparison Theorem. Let $f : [a,\infty[\to \mathbb{R}$ be bounded and integrable on $[a,b]$ $\forall b \in \mathbb{R}, b > a$.

1. If $\forall x \geq a : |f(x)| \leq g(x)$ and $g(x)$ is integrable on $[a,\infty[\implies f$ is integrable on $[a,\infty[$.
2. If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\,dx$ diverges $\implies \int_a^\infty f(x)\,dx$ diverges.

Hint. Sometimes an integral can be split into a normal integral and an improper integral:

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

McLaurin. Let $f : [1,\infty[\to [0,\infty[$ be mon. dec.

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \int_1^\infty f(x)\,dx \text{ converges.}$$

The following holds:

$$0 \leq \sum_{k=1}^{\infty} f(k) - \int_1^\infty f(x)\,dx \leq f(1)$$

Def. Let f be a function which is bounded and integrable on all intervals $[a+\epsilon, b]$ $\forall \epsilon > 0$. $f :]a,b] \to \mathbb{R}$ is integrable if $\lim\limits_{\epsilon\to 0} \int_{a+\epsilon}^b f(x)\,dx$ exists. In this case the limit is defined as $\int_a^b f(x)\,dx$. (The comparison theorem can be used for such integrals as well.)

6.9 Indefinite Integrals

Let $f : I \to \mathbb{R}$ be defined on the interval $I \subseteq \mathbb{R}$. If f is continuous there exists a primitive function F.

$$\int f(x)\,dx = F(x) + C$$

The indefinite integral is the inverse of the derivative.

Hint. $\int_{-\infty}^{\infty} f(x)\,dx = \lim\limits_{b\to-\infty} \int_b^a f(x)\,dx + \lim\limits_{c\to\infty} \int_a^c f(x)\,dx$.

$$\int\limits_{-\infty}^{\infty} f(x)\,dx \text{ conv.} \iff \int\limits_{-\infty}^{a} f(x)\,dx \text{ conv.} \wedge \int\limits_a^\infty f(x)\,dx \text{ conv.}$$

In general: Let $f :]a,b[\to \mathbb{R}$ such that it is integrable on each compact interval $[\tilde{a},\tilde{b}]$. Then

$$\int_a^b f(x)\,dx := \lim_{\tilde{a}\searrow a} \lim_{\tilde{b}\nearrow b} \int_{\tilde{a}}^{\tilde{b}} f(x)\,dx$$

6.10 Euler Gamma Function

Def. For $s > 0$:

$$\Gamma(s) := \int_0^\infty e^{-x} x^{s-1}\,dx.$$

The gamma function interpolates the function $n \mapsto (n-1)!$. It converges for all $s > 0$.

Useful Listings

Limits

$\lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to\infty} 1 + \frac{1}{x} = 1$

$\lim_{x\to\infty} e^x = \infty$

$\lim_{x\to-\infty} e^x = 0$

$\lim_{x\to\infty} e^{-x} = 0$

$\lim_{x\to-\infty} e^{-x} = \infty$

$\lim_{x\to\infty} \frac{e^x}{x^m} = \infty$

$\lim_{x\to-\infty} xe^x = 0$

$\lim_{x\to\infty} \ln(x) = \infty$

$\lim_{x\to0} \ln(x) = -\infty$

$\lim_{x\to\infty} (1+x)^{\frac{1}{x}} = 1$

$\lim_{x\to0} (1+x)^{\frac{1}{x}} = e$

$\lim_{x\to\infty} (1 \pm \frac{1}{x})^\lambda = 1$

$\lim_{x\to\infty} (1 + \frac{\lambda}{x})^x = e^\lambda$

$\lim_{x\to\infty} x^\lambda q^x = 0, \forall 0 \leq q < 1$

$\lim_{x\to\infty} \sqrt[x]{x} = 1$

$\lim_{x\to\pm\infty} (1 + \frac{1}{x})^x = e$

$\lim_{x\to\infty} (1 - \frac{1}{x})^x = \frac{1}{e}$

$\lim_{x\to\pm\infty} (1 + \frac{\lambda}{x})^{\alpha x} = e^{\lambda\alpha}$

$\lim_{x\to0} \frac{\sin(x)}{x} = 1$

$\lim_{x\to0} \frac{1}{\cos(x)} = 1$

$\lim_{x\to0} \frac{\cos(x)-1}{x} = 0$

$\lim_{x\to0} \frac{\log(1)-x}{x} = -1$

$\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$

$\lim_{x\to0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$

$\lim_{x\to0} \frac{e^x-1}{x} = 1$

$\lim_{x\to0} \frac{x}{\arctan(x)} = 1$

$\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

$\lim_{x\to\infty} (\frac{x}{x+\lambda})^x = e^{-\lambda}$

$\lim_{h\to0} (1 + \frac{h}{x})^{\frac{x}{h}} = e$

$\lim_{x\to0} \frac{\lambda^x-1}{x} = \ln(\lambda), \lambda > 0$

$\lim_{x\to0} \frac{e^{\lambda x}-1}{x} = \lambda$

$\lim_{x\to0} \frac{\ln(x+1)}{x} = 1$

$\lim_{x\to1} \frac{\ln(x)}{x-1} = 1$

$\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$

$\lim_{x\to\infty} \frac{\log(x)}{x^\lambda} = 0$

$\lim_{x\to\infty} \frac{\lambda x}{\lambda^x} = 0$

$\lim_{x\to\frac{\pi}{2}^-} \tan(x) = +\infty$

$\lim_{x\to\frac{\pi}{2}^+} \tan(x) = -\infty$

$\lim_{x\to0} x \log x = 0$

$\lim_{x\to0^+} x \ln x = 0$

Stirling Formula

$\lim_{x\to\infty} \frac{x!}{(\frac{x}{e})^x\sqrt{2\pi x}} = 1$

Series

- Geometric: $\sum_{n=0}^\infty q^n = \frac{1}{1-q}$ if $|q| < 1$
- Harmonic: $\sum_{n=1}^\infty \frac{1}{n}$ diverges
- Telescope: $\sum_{n=0}^\infty \frac{1}{n(n+1)} = 1$
- $\exp(z) := \sum_{n=0}^\infty \frac{z^n}{n!} = \lim_{n\to\infty}(1 + \frac{z}{n})^n = e^z$
- $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ converges $s > 1$ $(\frac{1}{1-\frac{1}{2^{s-1}}})$
- $\mathrm{p}(z) = \sum_{k=0}^\infty c_k z^k$ conv. abs. $|z| < \rho = \frac{1}{\limsup |c_k|^{1/k}}$

$\sum_{i=1}^n i = \frac{n(n+1)}{2}$ $\qquad \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$ $\qquad \sum_{i=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$

Taylor Series

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \mathcal{O}(x^5)\\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \mathcal{O}(x^7)\\
\sinh(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \mathcal{O}(x^7)\\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \mathcal{O}(x^6)\\
\cosh(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \mathcal{O}(x^6)\\
\tan(x) &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \mathcal{O}(x^7)\\
\tanh(x) &= x - \frac{x^3}{3} + \frac{2x^5}{15} - \mathcal{O}(x^7)\\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)\\
\sqrt{1+x} &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \mathcal{O}(x^4)
\end{aligned}$$

Parity of Functions

Even: $f(-x) = f(x) \quad \forall x \in D$ $\qquad |x|, \cos x, x^2$

Odd: $f(-x) = -f(x) \quad \forall x \in D$ $\qquad x, \sin, \tan, x^3$

Hint. Chaining odd functions results in an odd function.

Tips and Tricks from PVW script

(Inductive sequences) $a_1 := C,\ a_{n+1} := f(a_n)\ (\forall n > 1)$

1. Show monotonicity (either by induction over $\mathbb{N}$ or show $b_n = a_{n+1} - a_n$ (strictly) pos/neg $\forall n > 1$)
2. Show that the sequence is bounded (by induction)
3. Monotone convergence theorem $\implies \exists \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty} a_n = a \implies \forall$subseq. $l(n) : \lim_{n\to\infty} a_{l(n)} = a$
   Choose $l(n) = n + 1 \implies \lim_{n\to\infty} a_{n+1} = a$
   And thus $a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

Hint. Do no forget to write: f as composition of two continuous functions is continuous when using MVT.

(Change of variables) Let f and g be functions with f continuous in y_0 and g continuous in x_0 with $y_0 = \lim_{x\to x_0} g(x)$:

$$\lim_{x\to x_0} f(g(x)) = \lim_{y\to y_0} f(y)$$

(Powerrule) Let $f, g : D \to \mathbb{R}$ be cont. in x_0 with $\lim_{x\to x_0} f(x) = f(x_0) > 0$ and $\lim_{x\to x_0} g(x) = g(x_0)$ (both exist), then:

$$\lim_{x\to x_0} f(x)^{g(x)} = f(x_0)^{g(x_0)}$$

(Taylor Polynomial $sin(4)$ **with precision** d**)**

1. Choose a fitting center and denote the new function
   - $f(x) = \sin(x)$ with $x_0 = 4$, $a = \pi$ **or**
   - $f(x) = \sin(\pi + x)$ with $x_0 = 4 - \pi$, $a = 0$
2. Calculate the $(N + 1)$-th derivative of $f(x)$
3. Calculate the error $|R_n(f, x_0, a)|$
4. Solve the inequality $|R_n(f, x_0, a)| < d$ to N.
5. Solve the taylor polynomial for the given N.

Hint (Convergence of integrals)**.** An integral does not converge if it is not bounded in the given interval.

$$\lim_{x\to2} \frac{1}{(x-2)^2} = \infty \implies \int_0^\infty \frac{1}{(x-2)^2}\,dx \text{ diverges}$$

Trigonometry

Periodicity

$\sin(x) = \sin(x + 2\pi)$ $\qquad$ $\cos(x) = \cos(x + 2\pi)$

$\tan(x) = \tan(x + \pi)$ $\qquad$ $\cot(x) = \cot(x + \pi)$

Parity

$\sin(-x) = -\sin(x)$ $\qquad$ $\cos(-x) = \cos(x)$

$\tan(-x) = -\tan(x)$ $\qquad$ $\cot(-x) = -\cot(x)$

Complement

$\sin(\pi - x) = \sin(x)$ $\qquad$ $\cos(\pi - x) = -\cos(x)$

$\tan(\pi - x) = -\tan(x)$ $\qquad$ $\cot(\pi - x) = -\cot(x)$

Multiple-angles formulae

$\sin(2x) = 2\sin x \cos x$ $\qquad$ $\cos(2x) = \cos^2 x - \sin^2 x$

$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$ $\qquad$ $\cot(2x) = \frac{\cot x - \tan x}{2}$

$\sin(3x) = 3\sin x - 4\sin^3 x$ $\qquad$ $\cos(3x) = 4\cos^3 x - 3\cos x$

Addition Theorems

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

Multiplication

$$\sin x \sin y = \tfrac{1}{2}(\cos(x - y) - \cos(x + y))$$

$$\cos x \cos y = \tfrac{1}{2}(\cos(x - y) + \cos(x + y))$$

$$\sin x \cos y = \tfrac{1}{2}(\sin(x - y) + \sin(x + y))$$

Powers

$\sin^2 x = \frac{1-\cos(2x)}{2}$ $\qquad$ $\sin^3 x = \frac{3\sin x - \sin(3x)}{4}$

$\cos^2 x = \frac{1+\cos(2x)}{2}$ $\qquad$ $\cos^3 x = \frac{3\cos x + \cos(3x)}{4}$

$\tan^2 x = \frac{1-\cos(2x)}{1+\cos(2x)}$

Sum of functions

$$\sin x + \sin y = 2\sin\tfrac{x+y}{2}\cos\tfrac{x-y}{2}$$

$$\sin x - \sin y = 2\cos\tfrac{x+y}{2}\sin\tfrac{x-y}{2}$$

$$\cos x + \cos y = 2\cos\tfrac{x+y}{2}\cos\tfrac{x-y}{2}$$

$$\cos x - \cos y = 2\sin\tfrac{x+y}{2}\sin\tfrac{x-y}{2}$$

Miscellaneous

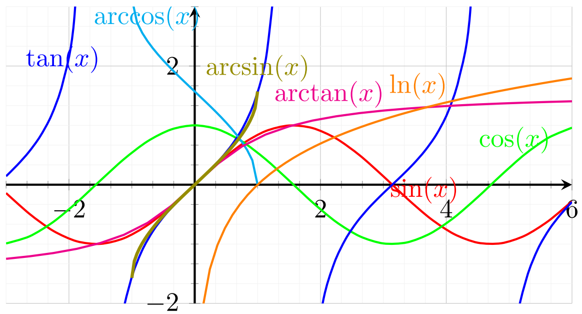
$\sin^2 x + \cos^2 x = 1$ $\qquad$ $\cosh^2 x - \sinh^2 x = 1$

$\sin x^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$ $\qquad$ $\cos x^{(n)} = \cos\left(x + \frac{n\pi}{2}\right)$

Angles

deg	0	30	45	60	90	120	135	150	180	270	360
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-$\frac{1}{2}$	-$\frac{1}{\sqrt{2}}$	-$\frac{\sqrt{3}}{2}$	-1	0	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-	-$\sqrt{3}$	-1	-$\frac{1}{\sqrt{3}}$	0	-	0

Important Functions



Useful bound for sin

$$\forall x \in \mathbb{R}_0^+ : \sin(x) \le x$$

Proof. Let $g(x) = x - \sin(x)$ with $g'(x) = 1 - \cos(x) \ge 0$ $\square$

Natural Logarithm Rules

$\ln(1) = 0$ $\qquad$ $\ln(e) = 1$

$\ln(xy) = \ln(x) + \ln(y)$ $\qquad$ $\ln(x/y) = \ln(x) - \ln(y)$

$\ln(x^y) = y \cdot \ln(x)$ $\qquad$ $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$

$(x^\alpha)^\beta = x^{\alpha\cdot\beta}$ $\qquad$ $\frac{x-1}{x} \le \ln(x) \le x - 1$

$\ln(1 + x^\alpha) \le \alpha x$ $\qquad$ $\log_\alpha(x) = \frac{\ln(x)}{\ln(\alpha)}$

Function Properties

For the following section consider an arbitrary function $f : X \to Y$.

Def (Well defined)**.** f is well defined if $f(x)$ exists $\forall x \in D$.

Def (Injective)**.** $\forall x, y \in X : f(x) = f(y) \implies x = y$
Proof strategies:

- Assume $f(x) = f(y)$ and then show that $x = y$
- Assume $x \neq y$ and show that $f(x) \neq f(y)$

Def (Surjective)**.** $\forall y \in Y\ \exists x \in X : f(x) = y$
Proof strategies:

- Take arbitrary $y \in Y$ and show that there is an element $x \in X$. Consider $f(x) = y$ and solve for x and check whether or not $x \in X$.

Def (Bijective)**.** f injective and surjective $\implies f$ bijective

(Show monotonicity) Calculate $f'(x)$ of $f : I \to \mathbb{R}$. If $\forall x \in I : f'(x) \le (\ge)\ 0 \implies f$ is monotone dec. (inc.)

Stirling Formula

$$n! \approx \frac{\sqrt{2\pi n}n^n}{e^n}$$

Can be used to approximate binomial coefficients e.g.

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Derivatives and Integrals (src: dcamenisch)

$\mathbf{F(x)}$	$\mathbf{f(x)}$
c	0
x^a	$a \cdot x^{a-1}$
$\frac{1}{a+1}x^{a+1}$	x^a
$\frac{1}{a\cdot(n+1)}(ax+b)^{n+1}$	$(ax+b)^n$
$\frac{x^{a+1}}{a+1}$	$x^a,\ a \neq -1$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sqrt[n]{x}$	$\frac{1}{n}x^{\frac{1}{n}-1}$
$\frac{2}{3}x^{\frac{3}{2}}$	$\sqrt{x}$
$\frac{n}{n+1}x^{\frac{1}{n}+1}$	$\sqrt[n]{x}$
e^x	e^x
$\ln(\lvert x\rvert)$	$\frac{1}{x}$
$\log_a(\lvert x\rvert)$	$\frac{1}{x\ln(a)} = \log_a(e^{\frac{1}{x}})$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$
$\cot(x) = \frac{\cos(x)}{\sin(x)}$	$\frac{1}{-\sin^2(x)}$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$\frac{-1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sinh(x) = \frac{e^x+e^{-x}}{2}$	$\cosh(x)$
$\cosh(x) = \frac{e^x-e^{-x}}{2}$	$\sinh(x)$
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$	$\frac{1}{\cosh^2(x)} = 1 - \tanh^2(x)$
$\frac{1}{f(x)}$	$\frac{-f'(x)}{(f(x))^2}$
a^{cx}	$a^{cx} \cdot c\ln(a)$
x^x	$x^x \cdot (1+\ln(x)),\ x>0$
$(x^x)^x$	$(x^x)^x(x+2x\ln(x)),\ x>0$
x^{x^x}	$x^{x^x}(x^{x-1} + \ln(x)\cdot x^x(1+\ln(x)))$

$\mathbf{F(x)}$	$\mathbf{f(x)}$
$\frac{1}{a}\ln(\lvert ax+b\rvert)$	$\frac{1}{ax+b}$
$\frac{ax}{c} - \frac{ad-bc}{c^2}\ln(\lvert cx+d\rvert)$	$\frac{ax+b}{cx+d}$
$\frac{1}{2a}\ln\left(\left\lvert\frac{x-a}{x+a}\right\rvert\right)$	$\frac{1}{x^2-a^2}$
$\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\ln(x+\sqrt{a^2+x^2})$	$\sqrt{a^2+x^2}$
$\frac{x}{2}\sqrt{a^2-x^2} - \frac{a^2}{2}\arcsin\left(\frac{x}{\lvert a\rvert}\right)$	$\sqrt{a^2-x^2}$
$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})$	$\sqrt{x^2-a^2}$
$\ln(x+\sqrt{x^2\pm a^2})$	$\frac{1}{\sqrt{x^2\pm a^2}}$
$\arcsin\left(\frac{x}{\lvert a\rvert}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$
$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$	$\frac{1}{x^2+a^2}$
$-\frac{1}{a}\cos(ax+b)$	$\sin(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$
$\frac{1}{a}\sin(ax+b)$	$\cos(ax+b)$
$\sin(ax+b)$	$a\cos(ax+b)$
$-\ln(\lvert\cos(x)\rvert)$	$\tan(x)$
$\ln(\lvert\sin(x)\rvert)$	$\cot(x)$
$\ln\left(\left\lvert\tan\left(\frac{x}{2}\right)\right\rvert\right)$	$\frac{1}{\sin(x)}$
$\ln\left(\left\lvert\tan(\frac{x}{2}+\frac{\pi}{4})\right\rvert\right)$	$\frac{1}{\cos(x)}$
$\frac{1}{2}(x-\sin(x)\cos(x))$	$\sin^2(x)$
$\frac{1}{2}(x+\sin(x)\cos(x))$	$\cos^2(x)$
$\frac{1}{4}(\frac{1}{3}\cos(3x)-3\cos(x))$	$\sin^3(x)$
$\frac{1}{4}(\frac{1}{3}\sin(3x)+3\sin(x))$	$\cos^3(x)$
$\tan(x)-x$	$\tan^2(x)$
$-\cot(x)-x$	$\cot^2(x)$
$x\arcsin(x)+\sqrt{1-x^2}$	$\arcsin(x)$
$x\arccos(x)-\sqrt{1-x^2}$	$\arccos(x)$
$x\arctan(x)-\frac{1}{2}\ln(1+x^2)$	$\arctan(x)$
$\ln(\cosh(x))$	$\tanh(x)$
$\ln(\lvert f(x)\rvert)$	$\frac{f'(x)}{f(x)}$

$\mathbf{F(x)}$	$\mathbf{f(x)}$
$x(\ln(\lvert x\rvert)-1)$	$\ln(\lvert x\rvert)$
$\frac{1}{n+1}(\ln x)^{n+1} \quad n\neq -1$	$\frac{1}{x}(\ln x)^n$
$\frac{1}{2n}(\ln x^n)^2 \quad n\neq 0$	$\frac{1}{x}\ln x^n$
$\ln(\lvert\ln(x)\rvert) \quad x>0, x\neq 1$	$\frac{1}{x\ln(x)}$
$\frac{1}{b\ln(a)}a^{bx}$	a^{bx}
$\frac{cx-1}{c^2}\cdot e^{cx}$	$x\cdot e^{cx}$
$\frac{1}{c}e^{cx}$	e^{cx}
$\frac{x^{n+1}}{n+1}\left(\ln(x)-\frac{1}{n+1}\right) \quad n\neq -1$	$x^n\ln(x)$
$\frac{e^{cx}(c\sin(ax+b)-a\cos(ax+b))}{a^2+c^2}$	$e^{cx}\sin(ax+b)$
$\frac{e^{cx}(c\cos(ax+b)+a\sin(ax+b))}{a^2+c^2}$	$e^{cx}\cos(ax+b)$
$\sin(x)\cos(x)$	$\frac{\sin^2(x)}{2}$
$\frac{1}{2}(f(x))^2$	$f'(x)f(x)$
$\sqrt{\pi}$	$\int_{-\infty}^{\infty} e^{-x^2}\,dx$
$\frac{1}{a(n+1)}(ax+b)^{n+1}$	$(ax+b)^n$
$\frac{(ax+b)^{n+2}}{(n+2)a^2} - \frac{b(ax+b)^{n+1}}{(n+1)a^2}$	$x(ax)^n$
$\frac{(ax^p+b)^{n+1}}{ap(n+1)}$	$(ax^p+b)^n x^{p-1}$
$\frac{1}{ap}\ln\lvert ax^p+b\rvert$	$(ax^p+b)^{-1}x^{p-1}$
$\frac{ax}{c} - \frac{ad-bc}{c^2}\ln\lvert cx+d\rvert$	$\frac{ax+b}{cx+d}$

Multiple Choice Questions

This section contains only the correct answers.

Sequences and Series

$\sum_{k=1}^{\infty}$ converges absolute and $\sum_{k=1}^{\infty}$ converges:

- $\diamond$ $\sum_{k=1}^{\infty} |a_k|^2$ converges **always** absolute.
- $\diamond$ $\sum_{k=1}^{\infty} a_k b_k$ converges **always** absolute.

Functions

Let f, g be monotonically increasing functions

$\not\Rightarrow$ $f \cdot g$ monotonically increasing

$\not\Rightarrow$ $\frac{f}{g}$ or $\frac{g}{f}$ monotonically increasing

$f : X \to Y, g : Y \to Z$ and $g \circ f : X \to Z$ bijective:

- $\diamond$ f is injective, g is surjective

Let $a, b \in \mathbb{R}$ with $a < b$ and $f :]a, b[\to \mathbb{R}$:

- $\diamond$ $|f|$ cont. $\not\Rightarrow$ f cont. (**CE** $f : -1\ x \le 0,\ 1\ x > 0$)
- $\diamond$ If f^2 and f^3 diff. in $]a, b[$ and $f(x) \neq 0 \forall x \implies f$ diff. since $\frac{h}{g}$ is diff. if both $h = f^3$ and $g = f^2$ are diff.

Let $f : [a, b] \to \mathbb{R}$ be a function and f_n a function sequence with converges uniformly to f:

- $\diamond$ If f_n for all $n \ge 1$ in $x_0 \in [a, b]$ bounded, then for all partitions P der limit of lower sums exists $\lim_{n\to\infty} \underline{S}(f_n, P) = \underline{S}(f, P)$
- $\diamond$ If f_n for all $n \ge 1$ in x_0 is continuous, then f is uniformly continuous
- $\diamond$ If f_n for all $n \ge 1$ in x_0 convex, then f is convex.
- $\diamond$ If f_n for all $n \ge 1$ is diff. in $x_0 \not\Rightarrow f$ is diff. in x_0

Let $f : [-1, 1] \to \mathbb{R}$ be an even, two times diff. function. Does $\forall x \in]0, 1[\ \exists \mu \in]0, x[$: $f(x) - f(0) = \frac{f''(\mu)x^2}{2}$ hold?

- $\diamond$ Yes. f' is diff. and cont. on $] - 1, 1[$. Thus using MVT $\forall x \in]0, 1[\ \exists \mu \in]0, x[$: $f'(x) - f'(0) = f''(\mu)(x - 0) \implies f'(x) = f''(\mu)x$. Thus $\int_0^x f'(x)\,dx = f''(\mu) \int_0^x x\,dx$. Using fundamental theorem of calculus, we have $f(x) - f(0) = \frac{f''(\mu)x^2}{2}$.

Let $g(x) = \begin{cases} \ln(x)\sin(2\pi x), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$ and let $g(n)_{n\ge1}$ be a sequence in $[0, 1]$ defined as

$$g_1 = \inf_{0\le t\le 1} g(t)$$

$$g_n = \begin{cases} \inf_{0\le t<\frac{1}{n}} g(t), & 0 \le x < \frac{1}{n} \\ \inf_{\frac{1}{n}\le t<\frac{2}{n}} g(t), & \frac{1}{n} \le x < \frac{2}{n} \\ \dots & \forall k \ge 2 \\ \inf_{\frac{j}{n}\le t<\frac{j+1}{n}} g(t), & \frac{j}{n} \le x < \frac{j+1}{n} \\ \dots & \\ \inf_{\frac{n-1}{n}\le t\le 1} g(t), & \frac{n-1}{n} \le x \le 1 \end{cases}$$

- $\diamond$ g is not smooth since $\lim_{x\to 0^+} \frac{g(x)-g(0)}{x} = -\infty$
- $\diamond$ $\exists x_0 \in]0, 1[$ with $g'(x_0) = 0$ since intermediate value theorem between $g'(\frac{1}{2}) > 0$ and $g'(\frac{3}{4}) < 0$.
- $\diamond$ The sequence $(g_n)_{n\ge1}$ converges uniformly to $g(x)$
- $\diamond$ $g(n)_{n\ge1}$ and g are integrable and $\int_0^1 g_n(x)\,dx \le \int_0^1 g(x)\,dx$ for each $n \ge 1$
- $\diamond$ $(g_n)_{n\ge1}$ and g are integrable and $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx = \int_0^1 g(x)\,dx$

General Exercises

Sequences and Series

Investigate if $\sum_{n=1}^{\infty} \log(\frac{n}{n+1})$ converges:

- $\diamond$ $\sum_{n=1}^{\infty} \log(\frac{n}{n+1}) = (\log(1)-\log(2))+(\log(2)-\log(3))+\dots$. This is a telescopic series and thus alle terms cancel out except for $\log(1) - \log(n + 1)$ and thus for $n \to \infty$ it the series must diverge.

Let $\sum_{k=1}^{\infty} a_k$ be abs. conv. and $\sum_{k=1}^{\infty} b_k$ conv. does $\sum_{k=1}^{\infty} b_k \sin(a_k)$ converge?

- $\diamond$ Proof $\lim_{k\to\infty} b_k = 0$ by using c_k where $c_1 = 0$ and $c_k = b_{k-1}$ for $k \ge 2$.
- $\diamond$ $\lim_{n\to\infty} \sum_{k=1}^{n} b_k = \lim_{n\to\infty} \sum_{k=1}^{n} c_k \implies$
  $\lim_{n\to\infty} b_k = \lim_{n\to\infty} (\sum_{k=1}^{n} b_k - \sum_{k=1}^{n} c_k) = 0$
- $\diamond$ Since $|\sin(x)| \le |x|$, then by comparison theorem $\sum_{k=1}^{\infty} \sin(a_k)$ converges absolutely.
- $\diamond$ Since b_k is bounded by constant C: $|b_n \sin(a_n)| \le C|a_n|$.

Calculate $\lim_{x\to 0} x^4 \sin(\frac{1}{x})$

- $\diamond$ $x \mapsto |\sin(\frac{1}{x})|$ is bounded by 1
- $\diamond$ $0 \le |x^4 \cdot \sin(\frac{1}{x})| \le x^4$
- $\diamond$ Both left and right handside functions have limit 0.
- $\diamond$ Thus by sandwich theorem it converges to 0.

Show that $\lim_{n\to\infty} \left(\cos\left(\frac{t}{\sqrt{n}}\right)^n = e^{-\frac{t^2}{2}}\right)$

- $\diamond$ $\cos(x) = 1 - \frac{x^2}{2} + \mathcal{O}(x^4)$ and $\log(1+x) = x + \mathcal{O}(x^2)$ hold $\forall t \in \mathbb{R}$ if $n \to \infty$

$$\begin{aligned} \cos\left(\frac{t}{\sqrt{n}}\right)^n &= \exp\left(n \log\left(\cos\left(\frac{t}{\sqrt{n}}\right)\right)\right) \\ &= \exp\left(n \log\left(1 - \frac{t^2}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)\right) \\ &= \exp\left(n\left(-\frac{t^2}{2n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right)\right) \\ &= \exp\left(-\frac{t^2}{2} + \mathcal{O}\left(\frac{1}{n}\right)\right) \\ &\underset{n\to\infty}{\to} \exp\left(-\frac{t^2}{2}\right) = e^{-\frac{t^2}{2}} \end{aligned}$$

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{\cos(x)-sin(x)^2}{4}$.

- $\diamond$ Show that $|f'(x)| \le \frac{3}{4}$ for all $\in \mathbb{R}$. Since $f'(x) = \frac{sin(x)(-1-2\cos(x))}{4} \implies |f'(x)| \le \frac{|-1|+|2\cos(x)|}{4} \le \frac{3}{4}$
- $\diamond$ Let $(x_n)_{n\ge1}$ be a sequence such that $x_1 = 0$ and $x_{n+1} = f(x_n)$. Show that $|x_{n+1} - x_n| \le (\frac{3}{4})^{n-1}\frac{1}{4}$. Since $x_2 = f(x_1) = f(0) = \frac{1}{4}$ we have $|x_2 - x_1| = \frac{1}{4}$. That means it hold for $n = 1$. Per induction we assume $|x_{n+1} - x_n| \le (\frac{3}{4})^{n-1}\frac{1}{4} \quad \forall n < N$. Because of the mean value theorem of diff. it exists a t between x_N and x_{N-1} such that $|x_{N+1} - x_N| = |f(x_N) - f(x_{N-1})|$ $\le f'(t)|x_N - x_{N-1}| \le \frac{3}{4}|x_N - x_{N-1}| \le (\frac{3}{4})^{N-1}\frac{1}{4}$.

- Show that x_n is a cauchy sequence. For $m \geq 0$ we have because of the triangle inequalitiy: $|x_{n+m} - x_n| \leq \sum_{k=1}^{m} |x_{n+k} - x_{n+k-1}| \leq \sum_{k=1}^{m} (\frac{3}{4})^{n+k-2}\frac{1}{4} = (\frac{3}{4})^{n-2}\frac{1}{4}\sum_{k=1}^{m}(\frac{3}{4})^k$
$\leq (\frac{3}{4})^{n-2}\frac{1}{4}(-1 + \sum_{k=0}^{\infty}(\frac{3}{4})^k) = (\frac{3}{4}^{n-2})\frac{1}{4}(-1 + \frac{1}{1-\frac{3}{4}})$
$= (\frac{3}{4})^{n-1}$ That means for a given $\epsilon > 0$ we choose $N(\epsilon) \in \mathbb{N}$, such that $(\frac{3}{4})^{n+1} < \epsilon \quad \forall n \geq N$. Then it holds that $|x_{n+m} - x_n| \leq \epsilon \quad \forall n \geq N, m \geq 0$. Which implies that it is indeed a cauchy sequence.
- Show that $x_n \to y$ for $n \to \infty$. Since the sequence is cauchy, it converges and because of continuity it follows $y = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(y)$.

Functions

Show that $f(x) = \begin{cases} x^x, & x > 0 \\ 1, & x = 0 \\ (-x)^{-x} & x < 0 \end{cases}$ is continuous.

- Because x^x is cont. for $x > 0$ and $(-x)^{-x}$ is cont. for $x < 0$ it suffices to prove $\lim_{x\to 0} f(x) = 1$.
- $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \overset{H}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = 0 \Rightarrow x^x \overset{\lim}{\to} 1$
- Since the function is even, we have $\lim_{x\to 0^-} x^x = 1$.

Show that $f(x) = \tan(x^3)$, $x \in]((-\frac{\pi}{2})^{\frac{1}{3}}), (\frac{\pi}{2})^{\frac{1}{3}})[$ is monotone and calculate its inverse:

- $f'(x) = \tan'(x^3)3x^2 = \frac{3x^2}{\cos(x^3)^2} > 0$ since $\cos(y)$ is strictly positive for $-\frac{\pi}{2} < y < \frac{\pi}{2}$. $x = 0$ does not break the monotonicity. $f^{-1}(x) = \arctan(x)^{\frac{1}{3}}$.

Calculate all cont. points of $f : \mathbb{R} \to \mathbb{R}$ with
$f(x) := \begin{cases} 1 - 2x, & x \in \mathbb{Q} \\ x - 3, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$:

- The graphs for intersect in only one point: $1 - 2x = x - 3 \iff x_0 = \frac{4}{3}$
- Proof continuity of x_0 with $y \in \mathbb{R}\setminus\mathbb{Q}$ by $|f(x_0) - f(y)| = |1 - \frac{8}{3} - y + 3| = |\frac{4}{3} - y| = |x_0 - y| < \delta$
- Set $\delta = \epsilon$ then it shows that f is continuous at x_0.
- To proof that it is the only continuous point in the function an each $x \in \mathbb{Q}$ lies between irrational numbers, we can take ϵ as the half of the distance between $f(x) = 1 - 2x$ and $x - 3$ and then it is clear that for δ small we have that if $y \in (x - \delta, x + \delta) \cap \mathbb{R} \setminus \mathbb{Q}$ then $|f(x) - f(y)| \geq \epsilon$. The existence of δ follows from the fact that the distance between the two lines varies continuously.

Derivatives and integrals

$f(x) = \int_x^2 \frac{e^{t^2}}{t}\,dt + \log(\frac{x}{2})$. Calculate $f'(x)$:

- Let $g(x) = \frac{e^{t^2}}{t}$ thus $f(x) = G(2) - G(x) + \log(\frac{x}{2})$
- It follows $f'(x) = G'(x) + \frac{1}{x}$ since $G(2) = C \in \mathbb{R}$
- Thus $f'(x) = g(x) - \frac{1}{x} = \frac{-e^{x^2}+1}{x}$

Let $f : [a,b] \to \mathbb{R}$, $G : [a,b] \to \mathbb{R}$ be integrable with G continuous and $F > 0$. Show that $c \in [a,b]$ exists such that $\int_a^b F(x)G(x)\,dx = G(c)\int_a^b F(x)\,dx$:

- Because $F > 0$:
$(\inf_{[a,b]} G)F(x) \leq G(x)F(x) \leq (\sup_{[a,b]} G)F(x)$ for all $x \in [a,b]$
- Thus by monotonicity of integrals:
$(\inf_{[a,b]} G)\int_a^b F(x)\,dx \leq \int_a^b G(x)F(x)\,dx \leq (\sup_{[a,b]} G)\int_a^b F(x)\,dx$
- because $F > 0$, it follows $\int_a^b F(x)\,dx > 0$ and thus we can write
$\inf_{[a,b]} G \leq \frac{\int_a^b G(x)F(x)\,dx}{\int_a^b F(x)\,dx} \leq \sup_{[a,b]} G$.
- Because G is continuous we can use the mean value theorem such that $\exists c \in [a,b] : G(c) = \frac{\int_a^b G(x)F(x)\,dx}{\int_a^b F(x)\,dx}$

Undergraduate approved theorems

\$ To-Ye † Cryptric ‡ ucinereo

†: $\cos(1) = 1$

‡: $e^0 = 0$

†, \$: $\frac{\ln(x)}{\ln(y)} = \ln(x - y)$

†, \$: $x^\alpha \cdot x^\beta = x^{\alpha\cdot\beta}$

‡: $x = \lambda \implies x^2 = |x^2| = \lambda^2 \implies |x| = \lambda$

\$: $\alpha x \cdot \alpha y = \alpha(x \cdot y)$

\$: $\cos(x^2) = \cos(x + \pi)$